A

TREATISE ON ALGEBRA,

43408

EMBRACING,

BESIDES THE ELEMENTARY PRINCIPLES, ALL THE HIGHER PARTS USUALLY TAUGHT IN

COLLEGES;

CONTAINING

MOREOVER, THE NEW METHOD OF CUBIC AND HIGHER EQUATIONS AS WELL AS THE DEVELOPMENT AND APPLICATION OF THE MORE RECENTLY DISCOVERED

THEOREM OF STURM.

BY GEORGE R. PERKINS, A. M.,

PROFESSOR OF MATHEMATICS IN NEW-YORK STATE NORMAL SCHOOL, AUTHOR OF "ELEMENTARY ARITHMETIC," "HIGHER ARITHMETIC," "ELEMENTS OF ALGEBRA," ETC., ETC.

SECOND EDITION, REVISED, ENLARGED AND IMPROVED.

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PREFACE

TO THE FIRST EDITION.

In presenting this volume to the Public, I would not claim to have unfolded many new principles of Algebra; I only claim to have judiciously combined and arranged principles already known. By commencing this work with the most elementary parts, and gradually ascending to the more complicated, I have designed to adapt it to the wants of students of every grade.

While I acknowledge, that, in general, the principles have long been known, I think I am justifiable in claiming some of the methods of operation as original.

This work will be found to contain, for the first time, I believe in any American school book, a demonstration and application of Sturm's Theorem; by the aid of which, we may at once determine the number of real roots, of any Algebraic Equation, with much more ease, than could be done by any previously discovered methods.

The method of finding the numerical values of the roots of cubic and higher equations, as fully explained under the last chapter, will, no doubt, be new to many, and interesting to all lovers of this science. It is particularly interesting on account of the ease with which it resolves itself into the method of extracting any root of a number, as explained in my Higher Arithmetic.

It would be extremely difficult to point out the exact sources from which I have drawn for this work, and even could I do so, these principles have been so long in use, we could not with safety say when, and with whom, they each originated. While I acknowledge the aid of many works on this science, I would give by far the greatest share of credit, to the eighth edition of Bourdon's most excellent treatise on Algebra.

Utica, July, 1842. GEORGE R. PERKINS.

PREFACE

TO THE SECOND EDITION.

THE present Edition has been very carefully Revised and considerably Enlarged. One entire Chapter on the subject of CONTINUED FRACTIONS, which are treated in quite a general manner, has been added. The subject of RECURRING SERIES has been re-written, and much simplified, and many other changes, which we deemed to be improvements, have been introduced.

Having almost daily made use of this work in my Classes, since its Publication, and always having had in view the changes which it would be desirable to make, in order to improve the work, we feel that we are now prepared to present the present edition as quite an improvement upon the first. It is believed it will be found to contain a pretty full and complete development of all the various subjects of Algebra, usually taught in our Colleges.

As we have already prepared a smaller work, especially designed for primary schools, it has been our aim to adapt this Treatise to the wants of the more advanced Schools and Colleges.

Utica, January, 1847. GEORGE R. PERKINS.

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CHAPTER I.

DEFINITIONS AND PRELIMINARY RULES.

DEFINITIONS.

(*Article* 1.) ALGEBRA is that branch of Mathematics, in which the calculations are performed by means of *letters* and *signs* or *symbols*.

(2.) In Algebra, quantities, whether given or required, are usually represented by letters. The first letters of the alphabet are, for the most part, used to represent known quantities; and the final letters are used for the unknown quantities.

(3.) The symbol $=$, is called the sign of *Equality*; and denotes that the quantities between which it is placed, are equal or equivalent to each other. Thus \$1 $=$ 100 cents, which is read, one dollar equals one hundred cents. Again, $a = b$, which is read, a equals b.

(4.) The symbol $+$, is called *plus*; and denotes that the quantities between which it is placed, are to be added together. Thus, $a + b = c$, which is read, a and b added, equals c. Again, $a + b + c = d + x$, which is read, a, b and c added, equals d added to x.

(5.) The symbol —, is called *minus;* and denotes that the quantity which is placed at the right of it is to be subtracted from the quantity on the left. Thus, $a - b = c$, which is read, a diminished by b equals c.

(6.) The symbol $\times$, is called the sign of *multiplication;* and denotes that the quantities between which it is placed are to be multiplied together. Thus, $a \times b = c$, which is read, a multiplied by b, equals c. Multiplication is also represented by placing a *dot* between the factors, or terms to be multiplied. Thus, $a \,.\, b$ is the same as $a \times b$. Another method, which is used as frequently as either of the above, is to unite the quantities in the form of a word. Thus, abc is the same as $a \times b \times c$, or $a \,.\, b \,.\, c$.

(7.) The symbol $\div$, is called the sign of *division;* and denotes that the quantity on the left of it is to be divided by the quantity on the right. Thus, $a \div b = c$, which is read, a divided by b equals c. Division is also indicated by placing the divisor under the dividend, with a horizontal line between them like a vulgar fraction. Thus, $\frac{x}{y}$ is the same as $x \div y$.

(8.) When quantities are enclosed in a parenthesis, brace, or bracket, they are to be treated as a simple quantity. Thus, $(a + b) \div c$, indicates that the sum of a and b is to be divided by c. Again, $(x - y) \div z = [x - y] \div z = \{x - y\} \div z$, each of which expressions is read, y subtracted from x and the remainder divided by z. The same thing may also be expressed by a bar or *vinculum*. Thus, $\overline{x - y} \div z$, which is read the same as the last three expressions.

(9.) The symbol $>$, is called the sign of *inequality;* and is used to express that the quantities between which it is placed are unequal. Thus, $b > a$ indicates that b is greater than a; and $b < c$ denotes that b is less than c.

(10.) When a quantity is added to itself several times, as $c + c + c + c$, we need write it but once, by placing before it a number to show how many times it has been taken. Thus, $c + c + c + c = 4c$. The number which is thus placed before the quantity is called the *coefficient* of the quantity. In the above example, 4 is the coefficient of c. A coefficient may consist, itself, of a letter. Thus, n is the coefficient of x in the expression nx; so also may x be regarded as the coefficient of n in the same expression.

(11.) The continued product of a quantity into itself is, usually, denoted by writing the quantity once, and placing a number over the quantity, a little to the right. Thus, $a \times a \times a$ is the same as a^3. The number thus placed over the quantity, is called the *exponent* of the quantity. Thus, 5 is the exponent of a in the expression a^5, and denotes that a is to be multiplied into itself, as a factor, five times.

(12.) When a quantity is multiplied continually into itself, the result is called a *power* of the quantity. Thus, a^6 is the *sixth* power of a, and a^3 is the *third* power of a, the exponent always indicating the degree of the power.

When a quantity is written without any exponent, it is understood that its exponent is a unit.

Thus, a is the same as a^1, and $(x + y) \times m$ is the same as $(x + y)^1 \times m^1$.

(13.) The symbol $\sqrt{\ }$, is called the *radical sign;* and denotes that a root of the quantity, over which it is placed, is to be extracted. Thus,

$\sqrt[2]{x}$, or simply $\sqrt{x}$, denotes the *square root* of x.

$\sqrt[3]{x}$ denotes the *cube root* of x.

$\sqrt[4]{x}$ denotes the *fourth root* of x.

The number placed over the radical is called the *index* of the root. Thus, 2, 3, and 4 are, respectively, the *indices* of the square root, cube root, and fourth root.

(14.) A root of a quantity may also be represented by means of a fractional exponent. Thus, the square root of a is $a^{\frac{1}{2}}$; the cube root of a is $a^{\frac{1}{3}}$; the fourth root of a is $a^{\frac{1}{4}}$; and so on for other roots.

By the same notation, $a^{\frac{2}{3}}$ is the cube root of the square of a, or the square of the cube root of a. For the same reason $a^{\frac{3}{5}}$ is the fifth root of the third power of a, or the third power of the fifth root of a.

(15.) The *reciprocal* of a quantity is a unit divided by that quantity. Thus, $\frac{1}{a}$ is the reciprocal of a; also $\frac{1}{3}$ is the reciprocal of 3.

(16.) The symbol $\therefore$, is equivalent to the phrase, *therefore*, or *consequently*.

(17.) When algebraic quantities are written without any sign prefixed, the sign *plus* is understood, and the quantities are said to be *positive* or *affirmative*; and those having the sign *minus* prefixed are called *negative* quantities. Thus, $a = +a$, $b = +b$, are each positive quantities; whilst $-a$, $-b$, are negative quantities. When the sign $-$, is prefixed to an isolated term, as $-a$, $-b$, it is not to be considered as a symbol of *operation*, but as a symbol of *condition*, merely showing that a and b are in a state or condition directly opposite to that denoted by $+a$ and $+b$. Thus, if the degrees of the thermometer above zero are called $+$, then those below must be called $-$.

(18.) An algebraic expression composed of two or more terms connected by $+$ or $-$, is called a *polynomial*. A polynomial composed of but two terms, is called a *binomial*; one composed of three terms, is called a *trinomial*.

Thus,

$$\left.\begin{array}{l} 3a + 46, \\ 7x^2 - 3y, \\ 3a^3 - x^2, \end{array}\right\} \text{are binomials.}$$

$$\left.\begin{array}{l} 3a^2 + 46 - x, \\ 4m - y + a, \\ 5g - x + y, \end{array}\right\} \text{are trinomials.}$$

(19.) Each of the literal factors which compose any term, is called a *dimension* of this term : the *degree* of a term is the number of the dimensions or factors. Thus,

$$\left.\begin{array}{l} 7a, \\ 5b, \end{array}\right\} \begin{array}{l}\text{are terms of one dimension, or of} \\ \text{the first degree.}\end{array}$$

$$\left.\begin{array}{l} 5ax, \\ 5xy, \end{array}\right\} \begin{array}{l}\text{are terms of two dimensions, or} \\ \text{of the second degree.}\end{array}$$

$$\left.\begin{array}{l} 7a^2b^3 = 7aabbb, \\ 3x^5 \quad = 3xxxxx, \end{array}\right\} \begin{array}{l}\text{are terms of five dimensions, or} \\ \text{of the fifth degree.}\end{array}$$

(20.) A polynomial is said to be *homogeneous*, when all its terms are of the same degree. Thus,

$$\left.\begin{array}{l} 3a - 5x + 2y, \\ b - y + m, \end{array}\right\} \begin{array}{l}\text{are homogeneous polynomials of} \\ \text{the first degree.}\end{array}$$

$$\left.\begin{array}{l} 4a^2 + 2x^2 - xy, \\ 7am - c^2 + a^2, \end{array}\right\} \begin{array}{l}\text{are homogeneous polynomials of} \\ \text{the second degree.}\end{array}$$

$$\left.\begin{array}{l} 5a^2b^3 - 6a^5 - 4x^4y, \\ 3a^4b - b^5 + 4a^3b^2, \end{array}\right\} \begin{array}{l}\text{are homogeneous polynomials} \\ \text{of the fifth degree.}\end{array}$$

(21.) Any combination of letters, by the aid of algebraic signs, is called an *algebraic expression*. Thus,

$$7x, \left\{ \begin{array}{l}\text{is an algebraic expression, denoting seven times} \\ \text{the quantity } x.\end{array}\right.$$

$$3\sqrt{b} + a^2, \left\{ \begin{array}{l}\text{is an algebraic expression, denoting that} \\ \text{the quantity } a \text{ is to be squared, and then} \\ \text{added to three times the square root of } b.\end{array}\right.$$

$$(a + b)^2, \left\{ \begin{array}{l}\text{is an algebraic expression, denoting that} \\ \text{the sum of } a \text{ and } b \text{ is to be squared.}\end{array}\right.$$

The algebraic expression $(a + b)(a - b) = a^2 - b^2$, is read, the sum of a and b, multiplied by the difference of a and b, equals the difference of the squares of a and b.

(22.) We will give some identical algebraic expressions, which may serve to exercise the student in reading algebraic formulas.

$$(a+b)^2 = a^2 + 2ab + b^2. \qquad (1)$$
$$(a-b)^2 = a^2 - 2ab + b^2. \qquad (2)$$
$$(a+b)^2 + (a-b)^2 = 2a^2 + 2b^2. \qquad (3)$$
$$\tfrac{1}{2}(a+b) + \tfrac{1}{2}(a-b) = a. \qquad (4)$$
$$\tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b) = b. \qquad (5)$$

Expression (1) is read, "the square of the sum of a and b is equal to the square of a, plus twice the product of a and b, plus the square of b."

Expression (2) is read, "the square of a diminished by b is equal to the square of a, minus twice the product of a and b, plus the square of b."

The student should read the remaining expressions for himself, and should also form other expressions, which he may in like manner translate into common language. He should also substitute particular values for a and b, in the above expressions, and see if the results on both sides oı the equations are identical.

Thus, the above expressions become, when

$$a = 2, \text{ and } b = 1.$$

$$(2+1)^2 = 4 + 4 + 1 = 9. \qquad (1)$$
$$(2-1)^2 = 4 - 4 + 1 = 1. \qquad (2)$$
$$(2+1)^2 + (2-1)^2 = 8 + 2 = 10. \qquad (3)$$
$$\tfrac{1}{2}(2+1) + \tfrac{1}{2}(2-1) = 2. \qquad (4)$$
$$\tfrac{1}{2}(2+1) - \tfrac{1}{2}(2-1) = 1. \qquad (5)$$

If $a = \frac{1}{2}$, and $b = \frac{1}{3}$, they will become

$$(\tfrac{1}{2} + \tfrac{1}{3})^2 = \tfrac{1}{4} + \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{25}{36}. \qquad (1)$$
$$(\tfrac{1}{2} - \tfrac{1}{3})^2 = \tfrac{1}{4} - \tfrac{1}{3} + \tfrac{1}{9} = \tfrac{1}{36}. \qquad (2)$$
$$(\tfrac{1}{2} + \tfrac{1}{3})^2 + (\tfrac{1}{2} - \tfrac{1}{3})^2 = \tfrac{1}{2} + \tfrac{2}{9} = \tfrac{13}{18}. \qquad (3)$$
$$\tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{3}) + \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{1}{2}. \qquad (4)$$
$$\tfrac{1}{2}(\tfrac{1}{2} + \tfrac{1}{3}) - \tfrac{1}{2}(\tfrac{1}{2} - \tfrac{1}{3}) = \tfrac{1}{3}. \qquad (5)$$

In this way the student should be exercised, until he becomes familiar with the nature of algebraic expressions.

ADDITION.

(23.) Addition, in Algebra, is finding the simplest expression for several algebraic quantities, connected by $+$ or $-$.

Suppose we wish to find the sum of

$$3a^2b + 7a^2b - 10a^2b + 4a^2b - 5a^2b - 2a^2b.$$

We first seek the sum of the positive quantities, by placing them under each other as in arithmetical addition, thus,

$$\begin{array}{l} +3a^2b \\ +7a^2b \\ +4a^2b \\ \hline +14a^2b = \text{sum of the positive terms.} \\ \hline \end{array}$$

Proceeding in the same way with the negative terms, we find

$$\begin{array}{l} -10a^2b \\ -\ \ 5a^2b \\ -\ \ 2a^2b \\ \hline -17a^2b = \text{sum of negative terms.} \\ \hline \end{array}$$

Therefore the total sum is $+14a^2b - 17a^2b = -3a^2b$.

We could proceed in a similar way for expressions of a like kind.

CASE I.

(24.) When the quantities are alike but have unlike signs, we have this

RULE.

I. *Place the different terms under each other, add the coefficients of the positive quantities into one sum, and the coefficients of the negative quantities into another.*

II. *Subtract the* LESS *from the* GREATER.

III. *Prefix the sign of the greater sum to the remainder, and annex the common letters.*

EXAMPLES.

1. What is the sum of

$$2abx - 7abx - 2abx + 12abx + abx - 3abx\,?$$

$$\begin{array}{r} 2abx \\ 12abx \\ abx \\ \hline 15abx \end{array} = \text{sum of positive terms.}$$

$$\begin{array}{r} -\ 7abx \\ -\ 2abx \\ -\ 3abx \\ \hline -12abx \end{array} = \text{sum of negative terms.}$$

Therefore $15abx - 12abx = 3abx =$ sum total.

2. What is the sum of

$$7amn - 3amn + 2amn + 5amn - 10amn\,?$$

Ans. amn.

3. What is the sum of

$$49axy - 37axy - 10axy + 100axy - 7axy + 4axy\,?$$

Ans. $99axy$.

4. What is the sum of

$$3\sqrt{ax} + 7\sqrt{ax} - 5\sqrt{ax} - 3\sqrt{ax} + 10\sqrt{ax} - 4\sqrt{ax}\,?$$

Ans. $8\sqrt{ax}$.

5. What is the sum of

$$4a\sqrt{b} - 5ab^{\frac{1}{2}} + 41a\sqrt{b} - 7ab^{\frac{1}{2}} + 10ab^{\frac{1}{2}}?$$

Ans. $43ab^{\frac{1}{2}}$.

6. What is the sum of

$$13a^{\frac{1}{2}}b^{\frac{1}{3}} + 6a^{\frac{1}{2}}\sqrt[3]{b} - 20a^{\frac{1}{2}}b^{\frac{1}{3}} + 10a^{\frac{1}{2}}b^{\frac{1}{3}}\,?$$

Ans. $9a^{\frac{1}{2}}b^{\frac{1}{3}}$.

7. What is the sum of

$$11a^4b^2\sqrt[4]{x} + a^4b^2x^{\frac{1}{4}} - 20a^4b^2x^{\frac{1}{4}} + 6a^4b^2\sqrt[4]{x}\,?$$

Ans. $-2a^4b^2\sqrt[4]{x}$.

CASE II.

(25.) When both quantities and signs are unlike, or some like and others unlike.

RULE.

I. *Find the sum of the like terms as in Case I.*

II. *Then write the sums one after another, with their proper signs.*

EXAMPLES.

1. What is the sum of

$$3ax - 2ab + 4xy - 2ax + 3xy + 7ab - 2xy + 6ax\,?$$

$$\begin{array}{r} 3ax \\ -2ax \\ 6ax \\ \hline 7ax \end{array} = \text{sum of the terms containing } ax.$$

$$\begin{array}{r} -2ab \\ 7ab \\ \hline 5ab \end{array} = \text{sum of the terms containing } ab.$$

$$\begin{array}{r} 4xy \\ 3xy \\ 2xy \\ \hline 5xy \end{array} = \text{sum of the terms containing } xy.$$

Therefore $7ax + 5ab + 5xy =$ total sum.

2. What is the sum of
$2a^2x - 3ax^2 + 2ab - 7a^2x + 4ax^2 - 8ab - 6a^2x + 10ax^2 + 12ab$?

$$\begin{array}{rrr} 2a^2x - & 3ax^2 + & 2ab \\ -7a^2x + & 4ax^2 - & 8ab \\ -6a^2x + & 10ax^2 + & 12ab \\ \hline -11a^2x + & 11ax^2 + & 6ab \end{array}$$

Ans. $-11a^2x + 11ax^2 + 6ab$

3.

$$\begin{array}{rrr} 3a^2b^3 - & 7ab^4 + & 5axy \\ -7a^2b^3 - & 2ab^4 - & axy \\ 8a^2b^3 + & ab^4 - & 7axy \\ a^2b^3 - & 10ab^4 + & 3axy \\ \hline 5a^2b^3 - & 18ab^4 & \end{array}$$

4.

$$\begin{array}{rrr} 4am - & 3am^2 - & 6ab \\ -7am + & 4am^2 + & ab \\ -8am - & 10am^2 - & 6ab \\ am + & am^2 + & 20ab \\ \hline -10am - & 8am^2 + & 9ab \end{array}$$

5.

$$\begin{array}{r}
7\sqrt{y}-4(a+m)\\
3\sqrt{y}-2(a+m)\\
\sqrt{y}+7(a+m)\\
5\sqrt{y}+\ \ (a+m)\\
\hline
16\sqrt{y}+2(a+m)\\
\hline
\end{array}$$

6.

$$\begin{array}{r}
4a^2+\ \ 5an\\
-3a^2-\ \ 7an\\
2a^2-\ \ 3an\\
5a^2+10an\\
\hline
8a^2+\ \ 5an\\
\hline
\end{array}$$

7.

$$\begin{array}{r}
a(a+b)+\ \ 3\sqrt{a-x}\\
-4a(a+b)-10\sqrt{a-x}\\
-7a(a+b)-\ \ 4\sqrt{a-x}\\
2a(a+b)+\ \ 6\sqrt{a-x}\\
\hline
-8a(a+b)-\ \ 5\sqrt{a-x}\\
\hline
\end{array}$$

8.

$$\begin{array}{r}
3x^{\frac{1}{2}}y+2m\sqrt{a}+\ \ 3\\
x^{\frac{1}{2}}y+7m\sqrt{a}-\ \ 7\\
13x^{\frac{1}{2}}y-\ \ m\sqrt{a}-10\\
-2x^{\frac{1}{2}}y+3m\sqrt{a}+\ \ 2\\
6x^{\frac{1}{2}}y-4m\sqrt{a}-\ \ 1\\
\hline
21x^{\frac{1}{2}}y-7m\sqrt{a}-13\\
\hline
\end{array}$$

9.

$$\begin{array}{r}
x^{\frac{1}{3}}m^{\frac{1}{2}}+\ \ x^{\frac{1}{2}}m^{\frac{1}{3}}\\
-11x^{\frac{1}{3}}m^{\frac{1}{2}}-\ \ 6x^{\frac{1}{2}}m^{\frac{1}{3}}\\
14x^{\frac{1}{3}}m^{\frac{1}{2}}+\ \ 5x^{\frac{1}{2}}m^{\frac{1}{3}}\\
-7x^{\frac{1}{3}}m^{\frac{1}{2}}-12x^{\frac{1}{2}}m^{\frac{1}{3}}\\
\hline
-3x^{\frac{1}{3}}m^{\frac{1}{2}}-12x^{\frac{1}{2}}m^{\frac{1}{3}}\\
\hline
\end{array}$$

10.

$$\begin{array}{r}
2b+\ \ 8x+6y\\
3b-\ \ 8x-4y\\
-7b-\ \ 3x-\ \ y\\
8b+10x+3y\\
9b-\ \ \ \ x-\ \ y\\
\hline
15b+\ \ 6x+3y\\
\hline
\end{array}$$

11. What is the sum of $3ag+6am-9xy+3ab-xy+4ag+10am-7xy-6ab+5xy+4ag-13am$?

Ans. $11ag+3am-12xy-3ab$.

12. What is the sum of $4a^2x - 5a^3y + 7am - 3a^2x - 10a^3y - 4am + 9a^3y - 7a^2x - 13am + 6a^3y - 11a^2x + am - a^3y + a^3y - 6a^2x$?

Ans. $-23a^2x - 9am$.

13. What is the sum of $-3xy + 5n + 3ax - 10am - 6xy + 7n - 4ax + 8am - 2xy + 10n + 6am - 4ax$?

Ans. $-11xy + 22n - 5ax + 4am$.

14. What is the sum of $4a^2 + 5a^2b^2c^2 - 9a^2 + 6a^2b^2c^2 + 10a^3x + 7a^3x + 8a^2 - 13a^2b^2c^2 + 5a^2 - 3a^3x + 3a^2b^2c^2$?

Ans. $8a^2 + a^2b^2c^2 + 14a^3x$.

15. What is the sum of $\sqrt{a} - 3xy - \sqrt[3]{m} - n + 6xy + 5\sqrt{a} + 7n - 7xy + 9\sqrt{a} - 7\sqrt[3]{m} + 16n - 5\sqrt{a}$?

Ans. $10\sqrt{a} - 4xy - 8\sqrt[3]{m} + 22n$.

16. What is the sum of $6a^{\frac{1}{3}}\sqrt{b} + 5x^2y^{\frac{1}{4}} - 7x + 5a^{\frac{1}{3}}b^{\frac{1}{2}} + 6x - 3a^{\frac{1}{3}}b^{\frac{1}{2}} - 4x^2\sqrt[4]{y} + 2a^{\frac{1}{3}}\sqrt{b} - 10x + 8a^{\frac{1}{3}}b^{\frac{1}{2}}$?

Ans. $18a^{\frac{1}{3}}b^{\frac{1}{2}} + x^2y^{\frac{1}{4}} - 11x$.

17. What is the sum of $3\sqrt{a+b} - 5\sqrt{x} + 5x^2y^3 + 7\sqrt{a+b} + 3x^2y^3 - 7\sqrt{x} + \sqrt{a+b} - 8x^2y^3 + 2\sqrt{x} + 10\sqrt{a+b}$?

Ans. $21\sqrt{a+b} - 10\sqrt{x}$.

18. What is the sum of $5a^3b^2c - 4ab^2c^3 + 2a^2b^2c^2 - 7ax + 6a^3b^2c - 5ab^2c^3 - 13ax - 7a^2b^2c^2 + 3ab^2c^3 - 8a^3b^2c$?

Ans. $3a^3b^2c - 6ab^2c^3 - 5a^2b^2c^2 - 20ax$.

SUBTRACTION.

(26.) SUBTRACTION, in Algebra, is the finding the simplest expression for the difference of two algebraic expressions.

If we subtract the positive quantity b from a, we obviously obtain

$$a - b,$$

which is the same as the addition of a and $-b$.

Again, if we wish to subtract $b - c$ from a, we obtain by subtracting b from a, $a - b$, but we have subtracted too much by the quantity c, therefore adding c, we get

$$a - b + c,$$

which is the same as the addition of a and $-b + c$.

From this, we see that subtracting a quantity is the same as adding it after the signs are changed.

Hence, for the subtraction of algebraic quantities we have this

RULE.

I. *Write the terms to be subtracted under the similar terms, if there are any, of those from which they are to be subtracted.*

II. *Conceive the signs of the terms of the polynomial to be subtracted, to be changed, and then proceed as in addition.*

EXAMPLES.

1.	2.
From $7ac - 3ab + d^2$	From $8amx - 4xy + 5y^2$
Take $4ac + 8ab + 4d^2$	Take $9amx + 10xy - 11y^2$
Rem. $3ac - 11ab - 3d^2$	Rem. $-amx - 14xy + 16y^2$

3.	4.
From $6xy - 3ac + 2m^3$	From $4a^3\sqrt{x} - 5a\sqrt[3]{y} + x$
Take $4xy - 7ac - 9m^3$	Take $3a^3\sqrt{x} + 3a\sqrt[3]{y} - 7x$
Rem. $2xy + 4ac + 11m^3$	Rem. $a^3\sqrt{x} - 8a\sqrt[3]{y} + 8x$

5. From $3a^2bc - 7axy + 3my + a$ take $a^2bc + 8axy + 6a - 4my$.

Ans. $2a^2bc - 15axy + 7my - 5a$.

6. From $8ab\sqrt{c} - 12a^3b + 6cx - 7xy$ take $9ab\sqrt{c} - 13a^3b + 8xy - an + 3cx$.

Ans. $-ab\sqrt{c} + a^3b + 3cx - 15xy + an$.

7. From $15a^3x - 14a^2y + 3ab^5 + 6amn$ take $6mg + 3a - 5a^3x - 7a^2y + 3ab^5 - 4amn - 4$.

Ans. $20a^3x - 7a^2y + 10amn - 6mg - 3a + 4$.

8. From $13a^4x^3y + 3ax - 7ab + 6mg - x^2y^2$ take $5xy + 4a^4x^3y - 6ax + 9ab + 2mg + 5x^2y^2$.

Ans. $9a^4x^3y + 9ax - 16ab + 4mg - 6x^2y^2 - 5xy$.

9. From $7a^{\frac{1}{2}}b^{\frac{1}{3}} + 3a^2 - 4b^4x - 3axy + 4x^2 - 3xy^3$ take $3ab - 17 + 4a^2 - 5a^{\frac{1}{2}}b^{\frac{1}{3}} - 7b^4x + 3xy^3$.

Ans. $12a^{\frac{1}{2}}b^{\frac{1}{3}} - a^2 + 3b^4x - 3axy + 4x^2 - 6xy^3 - 3ab + 17$.

10. From $4a^3bx - 7axy + 3p^2q + 17 - x$ take $4p^2q - 13 + 7a^3bx + 8axy - 7x + 3f - mg + n$.

Ans. $-3a^3bx - 15axy - p^2q + 30 + 6x - 3f + mg - n$.

11. From $6am + x$ take $3am + y$.

Ans. $3am + x - y$.

12. From $3a^2m - 6x^2y^3 + 2xy$ take $4a^2m + 6x^2y^3 + 5xy$.

Ans. $-a^2m - 12x^2y^3 - 3xy$.

13. From $3amx - 43 + x - y + 27d$ take $15n + 7g - 3 + 4y - 8d + 7amx - x + pq - rs$.

Ans. $-4amx - 40 + 2x - 5y + 35d - 15n - 7g - pq + rs$.

14. From $a + b$ take $a - b$.

Ans. $2b$.

(27.) We can express the subtraction of one polynomial from another, by writing the polynomial which is to be subtracted, after enclosing it within a parenthesis, immediately after the other polynomial from which it is to be subtracted, observing to place the negative sign before the parenthesis.

Thus, $ab - 6xy + 3am - (4ab + 3xy + am)$

denotes, that the polynomial enclosed within the parenthesis is to be subtracted from the one which precedes it; and since, by (Art. 26), to perform subtraction, we must change all the signs of the terms to be subtracted, we may remove the parenthesis provided we change the signs of the terms which it encloses: and conversely, we may enclose any number of terms within the parenthesis, with a negative sign before it, if we observe to change the signs of the terms thus enclosed.

In this way we can transform the expression

$$a^2b + xy - 7am - (mx + 6 - 13x^2),$$

into

$$a^2b + xy - 7am - mx - 6 + 13x^2,$$

into

$$a^2b + xy - (7am + mx + 6 - 13x^2),$$

into

$$a^2b - (-xy + 7am + mx + 6 - 13x^2),$$

into

$$a^2b + xy - 7am - mx - (6 - 13x^2).$$

MULTIPLICATION.

(28.) If we wish to multiply a by b, we must repeat a as many times as there are units in b, which, by (Art. 6), is done by writing b immediately after a, thus, a multiplied by $b = ab$.

Again, if we wish to multiply a by $-b$, we observe that this is the same as to multiply $-b$ by a, hence we must repeat $-b$ as many times as there are units in a: repeating a minus quantity once, twice, thrice, or any number of times can not change it to a positive quantity. Hence, $-b$ multiplied by a, or, which is the same, a multiplied by $-b = -ab$.

Finally, if we wish to multiply $a - b$ by $c - d$, we will first multiply $a - b$ by c, we thus obtain

$$\begin{array}{l} a - b \\ c \\ \hline ac - bc \text{ for } a - b \text{ repeated } c \text{ times.} \end{array}$$

This result is evidently too great by the product of $a - b$ by d, since it was required to repeat $a - b$ as many times as there are units in c less d.

Then repeating $a - b$ as many times as there are units in d, we have

$$\begin{array}{l} a - b \\ d \\ \hline ad - bd \text{ for } a - b \text{ repeated } d \text{ times.} \end{array}$$

Subtracting this last result from the former, we have $ac - bc - (ad - bd)$, which, by (Art. 27), becomes

$ac - bc - ad + bd$ for the product of $a - b$ by $c - d$.

Hence, we see that $-b$, when multiplied by $-d$, produces the product $+bd$

If we wished to multiply a by $-b$, it would hardly be correct to say, that we are to repeat a *minus* b times; for a quantity cannot be repeated a *minus* number of times. But when we wish to multiply a by $-b$, we evidently wish to repeat a as many times as there are units in b, and then to give to the product the negative sign; that is, when the multiplier is negative, we must multiply as though it were positive, and then give to the product a contrary sign.

Applying this principle to the case of $-a$ multiplied by $-b$. We know that $-a$ multiplied by $+b$ gives $-ab$ for the product; therefore $-a$ multiplied by $-b$ must give the same product taken with a contrary sign; that is, $-a$ multiplied by $-b$ must give $+ab$.

(29.) From all this, we discover, *that the product will have the sign* $+$, *when both factors have like signs, and the product will have the sign* $-$, *when the factors have contrary signs.*

If we wish to multiply $3a^2b$ by $4a^3b^2$, we observe that

$$3a^2b = 3aab$$
$$4a^3b^2 = 4aaabb$$

Hence, the product will be

$$3aab \times 4aaabb = 12aaaaabbb = 12a^5b^3.$$

Here we discover that the exponent of a, in the product, is equal to the sum of the exponents of a in the factors; likewise the exponent of b, in the product, is equal to the sum of the exponents of b in the factors.

(30.) *Hence the product of several letters of different exponents is equal to the product of all the letters, having for exponents the sums of their respective exponents in the factors.*

CASE I.

(31.) From what has been said, we have, for multiplying together two monomials, this

RULE.

I. *Multiply the coefficients, observing to prefix the sign + when both factors have like signs; and the sign — when they have contrary signs.*

II. *Write the letters one after another; if the same letter occur in both factors, add the exponents for a new exponent.*

(32.) The product will be the same in whatever order the letters are placed, but it will be found more convenient, in practice, to have a uniform order for their arrangement. The order usually adopted is to place them alphabetically.

EXAMPLES.

1. Multiply $11ax^3y$ by $3a^4y^3$.

Ans. $33a^5x^3y^4$.

2. What is the product of $3am^2$ by $6a^2b^3x$?

Ans. $18a^3b^3m^2x$.

3. What is the product of $10c^4d^5$ by $9a^5cd$?

Ans. $90a^5c^5d^6$.

4. Multiply $-13ac^3$ by $-4a^3b^6c^2$.

Ans. $52a^4b^6c^5$.

5. Multiply $a^mb^pc^q$ by a^nb^r.

Ans. $a^{m+n}b^{p+r}c^q$.

6. Multiply $-17x^3y$ by $3xyz$.

Ans. $-51x^4y^2z$.

7. Multiply $\frac{3}{4}ab^3cd$ by $\frac{1}{7}axy$.

Ans. $\frac{3}{28}a^2b^3cdxy$.

8. Multiply $-\frac{1}{2}xyz$ by $\frac{1}{2}x^2y^3z^4$.

Ans. $-\frac{1}{4}x^3y^4z^5$.

9. Multiply $7m^9n^8p^7$ by $6mn^2p^3$.

Ans. $42m^{10}n^{10}p^{10}$.

CASE II.

(33.) Polynomials may be multiplied together by the following

RULE.

I. *Multiply all the terms of the multiplicand successively by each term of the multiplier, and observe the same rules for the signs and exponents as in Case I.*

II. *When there arise several partial products alike, they must be placed under each other, and then added together in the total product.*

(34.) The total product will be the same in whatever order we multiply by the terms of the multiplier, but for the sake of order and uniformity, we begin with the left-hand term.

EXAMPLES.

1. What is the product of $3a^2-6ax+y$ by $3a-m$?

OPERATION.

$$\begin{array}{l} 3a^2-6ax+y \\ 3a-m \\ \hline \end{array}$$

Ans. $9a^3-18a^2x+3ay-3a^2m+6amx-my$.

2. What is the product of $6x^2-3y^3+a$ by x^2-2y^3-a?

OPERATION.

$$\begin{array}{l}
6x^2-3y^3+a \\
x^2-2y^3-a \\
\hline
6x^4-3x^2y^3+ax^2 \\
\qquad -12x^2y^3-6ax^2+6y^6-2ay^3 \\
\qquad\qquad\qquad\qquad\qquad +3ay^3-a^2 \\
\hline
\text{Ans. } 6x^4-15x^2y^3-5ax^2+6y^6+ay^3-a^2.
\end{array}$$

3. What is the product of b^2m-3ay by $6x-3$?

Ans. $6b^2mx-18axy-3b^2m+9ay$.

4. What is the product of $7l-2m-9$ by $3l-11m$?

Ans. $21l^2-83lm-27l+22m^2+99m$.

5. Multiply $2a+5b+3c-5e$ by $3a+10b+15f$.

Ans. $\begin{cases} 6a^2+35ab+9ac-15ae+50b^2+30bc \\ -50be+30af+75bf+45cf-75ef. \end{cases}$

6. Multiply $a+b+c+d$ by $a-b-c-d$.

Ans. $a^2-b^2-2bc-2bd-c^2-2cd-d^2$.

7. Multiply $a^6+a^4+a^2$ by a^2-1.

Ans. a^8-a^2.

8. Multiply a^2+az+z^2 by a^2-az+z^2.

Ans. $a^4+a^2z^2+z^4$.

9. Multiply $a+b$ by $a+b$.

Ans. $a^2+2ab+b^2$.

10. Multiply $a-b$ by $a-b$.

Ans. $a^2-2ab+b^2$.

11. Multiply $a+b$ by $a-b$.

Ans. a^2-b^2.

(35.) The last three examples, when translated into common language, give three distinct and important theorems, which we will proceed to illustrate.

Example 9 is the same as

$$(a+b)\times(a+b)=(a+b)^2=a^2+2ab+b^2;$$

which, when translated, gives

THEOREM I.

The square of the sum of two quantities is the same as the square of the first, plus twice the product of both, plus the square of the second.

EXAMPLES.

1. $(x+y)\times(x+y)=(x+y)^2=x^2+2xy+y^2$.
2. $(2x+a)\times(2x+a)=(2x+a)^2=4x^2+4ax+a^2$.
3. $(5m+3)\times(5m+3)=(5m+3)^2=25m^2+30m+9$.

Example 10 is the same as

$$(a-b)\times(a-b)=(a-b)^2=a^2-2ab+b^2;$$

which, when translated, gives

THEOREM II.

The square of the difference of two quantities is equal to the square of the first, minus twice the product of both, plus the square of the second.

EXAMPLES.

1. $(x-y)\times(x-y)=(x-y)^2=x^2-2xy+y^2$.
2. $(3a-b)\times(3a-b)=(3a-b)^2=9a^2-6ab+b^2$.
3. $(5a-x)\times(5a-x)=(5a-x)^2=25a^2-10ax+x^2$.

Example 11 is the same as

$$(a+b)\times(a-b)=a^2-b^2;$$

which, when translated, gives

THEOREM III.

The sum of two quantities multiplied by their difference, is equal to the square of the greater, minus the square of the less.

EXAMPLES.

1. $(x+y)\times(x-y)=x^2-y^2$.
2. $(3a+b)\times(3a-b)=9a^2-b^2$.
3. $(7m+y)\times(7m-y)=49m^2-y^2$.

DIVISION.

(36.) We know by the principles of Arithmetic, that, if, in Division, we multiply the divisor into the quotient, the product will be the dividend.

Therefore, referring to what has been said under Multiplication (Art. 29), we infer that when the dividend has the sign +, the divisor and quotient must have the same sign; but when the dividend has the sign —, then the divisor and quotient must have contrary signs.

(37.) *Hence, when the dividend and divisor have like signs, the quotient will have the sign +; and when the dividend and divisor have contrary signs, the quotient will have the the sign —.*

We have also seen under Multiplication (Art. 30), that the product of several letters of different exponents is equal to the product of all the letters with the sum of their respective exponents for new exponents.

(38.) *Hence, to divide any power of a letter by a different power of the same letter, it is obvious that the quotient will be a power of the same letter, having for exponent the excess of the exponent in the dividend above that of the divisor.*

(39.) If we divide continually the expression

$a^5 = aaaaa$ by a, we shall find

$$a^5 \div a = a^{5-1} = a^4 = aaaa;$$
$$a^4 \div a = a^{4-1} = a^3 = aaa;$$
$$a^3 \div a = a^{3-1} = a^2 = aa;$$
$$a^2 \div a = a^{2-1} = a^1 = a;$$
$$a^1 \div a = a^{1-1} = a^0 = 1;$$

$$a^0 \div a = a^{0-1} = a^{-1} = \frac{1}{a} = \text{reciprocal of } a;$$

$$a^{-1} \div a = a^{-1-1} = a^{-2} = \frac{1}{aa} = \frac{1}{a^2} = \text{reciprocal of } a^2;$$

$$a^{-2} \div a = a^{-2-1} = a^{-3} = \frac{1}{aaa} = \frac{1}{a^3} = \text{reciprocal of } a^3;$$

$$a^{-3} \div a = a^{-3-1} = a^{-4} = \frac{1}{aaaa} = \frac{1}{a^4} = \text{reciprocal of } a^4;$$

&c. &c.

(40.) *From the above scheme, we see, that whenever the exponent of a quantity becomes* 0, *its value is reduced to* 1.

(41.) *That whenever it is negative, it is the reciprocal of what it would be were it positive.*

(42.) *Hence, changing the sign of the exponent of a quantity is the same as taking its reciprocal.*

CASE I.

(43.) From what has been said, we have, for dividing one monomial by another, this

RULE.

I. *Divide the coefficient of the dividend by that of the divisor, observing to prefix to the quotient the sign* $+$ *when the signs of the dividend and divisor are alike, and the sign* $-$ *when they are contrary.*

II. *Subtract the exponents of the letters in the divisor from the exponents of the corresponding letters of the dividend; if letters occur in the divisor which do not in the dividend, they may* (Art. 42) *be written in the quotient by changing the signs of their exponents.*

(44.) It must be recollected here, and in all cases hereafter, that when the exponent of a letter is not written, 1 is

always understood (Art. 12); and when the exponent is 0, the value of the power is 1. (Art. 40.)

EXAMPLES.

1. What is the quotient of $14a^2x^3$ divided by $7ax^3y$?

Dividing the coefficients we find 2, to which if we annex the letters after subtracting the exponents. we have

$$2a^{2-1}x^{3-3}y^{-1}=2ay^{-1},$$

the x has disappeared, since its exponent became 0, and its value therefore was 1, by (Art. 40.) And since the y occurred in the divisor and not in the dividend, it was written in the quotient with the sign of the exponent changed. (Art. 42.)

2. What is the quotient of $35a^2b^3c$ divided by $5abc$?

Ans. $7ab^2$.

3. What is the quotient of $-44mnx^2$ divided by $22\,abcx$?

Ans. $-2a^{-1}b^{-1}c^{-1}mnx$.

4. Divide $-7x^2y$ by $10x^3y$.

Ans. $-\frac{7}{10}x^{-1}$.

5. Divide $3a^5m^4n^6$ by $-6amn$.

Ans. $-\frac{1}{2}a^4m^3n^5$.

6. Divide $35x^3yz^5$ by $-7x^2y^3z^5$.

Ans. $-5xy^{-2}$.

7. Divide cd^{11} by $-13cd^{12}$.

Ans. $-\frac{1}{13}d^{-1}$.

8. Divide $-3a^mb^n$ by $-4a^rb\,c^r$.

Ans. $\frac{3}{4}a^{m-r}b^{n-1}c^{-r}$.

9. Divide $-17a^6l^5m^{-1}$ by $-4a^{-1}l^4m^{-6}$.

Ans. $\frac{17}{4}a^7lm^5$.

10. Divide $13x^{-3}y^{-7}$ by $-26xy$.

Ans. $-\frac{1}{2}x^{-4}y^{-8}$.

(45.) To divide one polynomial by another, we shall imitate the arithmetical method of *long division*. And in the arrangement of the work we shall follow the *French* method of placing the divisor at the right of the dividend. Thus, to divide

$$a^3+a^2x+ab+bx \text{ by } a+x,$$

we proceed as follows:

OPERATION.

$$\begin{array}{r|l}
\text{Dividend} = a^3 + a^2x + ab + bx & a + x = \text{divisor} \\
a^3 + a^2x & \overline{a^2 + b = \text{quotient}} \\
\hline
ab + bx & \\
ab + bx & \\
\hline
0 &
\end{array}$$

Having placed the divisor at the right of the dividend, we seek how many times its left-hand term is contained in the left-hand term of the dividend, which we find to be a^2, which we place directly under the divisor, and then multiply the divisor by it, and subtract the product from the dividend; then bringing down the remaining terms, we again seek how many times the left-hand term of the divisor is contained in the left-hand term of this remainder, which we find to be b; we then multiply the divisor by b, and again subtracting there remains nothing; so that a^2+b is the complete quotient.

That the operation may be the most simple, it will be necessary to arrange both dividend and divisor according to the powers of some particular letter, commencing with the highest power.

CASE II.

(46.) To divide one polynomial by another, we have this

RULE.

I. *Arrange the dividend and divisor with reference to a certain letter; then divide the first term on the left of the dividend by the first term on the left of the divisor, the result is the first term of the quotient; multiply the divisor by this term, and subtract the product from the dividend.*

II. *Then divide the first term of the remainder by the first term of the divisor, which gives the second term of the quotient; multiply the divisor by this second term, and subtract the product from the result after the first operation. Continue this process until we obtain 0 for remainder; or when the division does not terminate, which is frequently the case, we can carry on the above process as far as we choose, and then place the last remainder over the divisor, forming a fraction, which must be added to the quotient.*

EXAMPLES.

1. What is the quotient of $2a^2b + b^3 + 2ab^2 + a^3$ divided by $a^2 + b^2 + ab$?

Arranging the terms according to the powers of a, and operating agreeably to the above rule, we have

OPERATION.

$$\begin{array}{rl|l}
\text{Dividend} = & a^3 + 2a^2b + 2ab^2 + b^3 & a^2 + ab + b^2 = \text{divisor} \\
& a^3 + \ \ a^2b + \ \ ab^2 & \overline{a + b = \text{quotient}} \\
\hline
& a^2b + \ \ ab^2 + b^3 & \\
& a^2b + \ \ ab^2 + b^3 & \\
\hline
& 0 &
\end{array}$$

2. What is the quotient of $a^2b-3a^2+2ab-6a-4b+22$ divided by $b-3$?

OPERATION.

$$
\begin{array}{l|l}
a^2b-3a^2+2ab-6a-4b+22 & b-3 \\
\cline{2-2}
\underline{a^2b-3a^2} & a^2+2a-4+\dfrac{10}{b-3} \\
\qquad 2ab-6a & \\
\qquad \underline{2ab-6a} & \\
\qquad\qquad -4b+22 & \\
\qquad\qquad \underline{-4b+12} & \\
\qquad\qquad\qquad 10=\text{remainder.} &
\end{array}
$$

3. Divide $x^6-x^4+x^3-x^2+2x-1$ by x^2+x-1.

OPERATION.

$$
\begin{array}{l|l}
x^6-x^4+x^3-x^2+2x-1 & x^2+x-1 \\
\cline{2-2}
\underline{x^6+x^5-x^4} & x^4-x^3+x^2-x+1. \\
\quad -x^5+x^3-x^2 & \\
\quad \underline{-x^5-x^4+x^3} & \\
\qquad x^4-x^2+2x & \\
\qquad \underline{x^4+x^3-x^2} & \\
\qquad\quad -x^3+2x-1 & \\
\qquad\quad \underline{-x^3-x^2+x} & \\
\qquad\qquad x^2+x-1 & \\
\qquad\qquad \underline{x^2+x-1} & \\
\qquad\qquad\qquad 0 &
\end{array}
$$

4. What is the quotient of $x^3-3ax^2+3a^2x-a^3$ divided by $x-a$?

OPERATION.

$$\begin{array}{l|l} x^3 - 3ax^2 + 3a^2x - a^3 & x - a \\ \cline{2-2} x^3 - \ \ ax^2 & x^2 - 2ax + a^2 \\ \cline{1-1} \quad - 2ax^2 + 3a^2x & \\ \quad - 2ax^2 + 2a^2x & \\ \cline{1-1} \qquad\qquad a^2x - a^3 & \\ \qquad\qquad a^2x - a^3 & \\ \cline{1-1} \qquad\qquad\qquad 0 & \end{array}$$

5. Divide $14af - 21bf + 7cf + 6ag - 9bg + 3cg$ by $7f + 3g$. Ans. $2a - 3b + c$.

6. Divide $4x^3 + 4x^2 - 29x + 21$ by $2x - 3$.
Ans. $2x^2 + 5x - 7$.

7. Divide $119c^2 - 200cd + 408ce - 113ch - 39d^2 + 72de + 37dh - 96eh + 20h^2$ by $17c + 3d - 4h$.
Ans. $7c - 13d + 24e - 5h$.

8. Divide $72x^4 - 78x^3y - 10x^2y^2 + 17xy^3 + 3y^4$ by $6x^2 - 4xy - y^2$.
Ans. $12x^2 - 5xy - 3y^2$.

9. Divide $36a^2b - 63ab^2 + 20b^3$ by $12ab - 5b^2$.
Ans. $3a - 4b$.

10. Divide $a^2 - b^2$ by $a - b$.
Ans. $a + b$.

11. Divide $a^4 - b^4$ by $a - b$.
Ans. $a^3 + a^2b + ab^2 + b^3$.

(47.) The following examples cannot be accurately performed, there being still a remainder, however far the division be carried.

12. Dividing 1 by $1-b$, we have in succession

$$1 \div (1-b) = 1 + \frac{b}{1-b}$$
$$= 1 + b + \frac{b^2}{1-b}$$
$$= 1 + b + b^2 + \frac{b^3}{1-b}$$
$$= 1 + b + b^2 + b^3 + \frac{b^4}{1-b}$$

&c. &c.

13. $$1 \div (1+b) = 1 - \frac{b}{1+b}$$
$$= 1 - b + \frac{b^2}{1+b}$$
$$= 1 - b + b^2 - \frac{b^3}{1+b}$$
$$= 1 - b + b^2 - b^3 + \frac{b^4}{1+b}$$

&c. &c.

14. $$(1+x) \div (1-x) = 1 + \frac{2x}{1-x}$$
$$= 1 + 2x + \frac{2x^2}{1-x}$$
$$= 1 + 2x + 2x^2 + \frac{2x^3}{1-x}$$
$$= 1 + 2x + 2x^2 + 2x^3 + \frac{2x^4}{1-x}$$

&c. &c.

CHAPTER II.

ALGEBRAIC FRACTIONS.

(48.) In our operations upon algebraic fractions, we shall follow the corresponding operations upon numerical fractions, so far as the nature of the subject will admit.

CASE I.

To reduce a monomial fraction to its lowest term, we have this

RULE.

I. *Find the greatest common measure of the coefficients of the numerator and denominator.* (See Arithmetic.)

II. *Then, to this greatest common measure annex the letters which are common to both numerator and denominator, give to these letters the lowest exponent which they have, whether in the numerator or denominator: the result will be the greatest common measure of both numerator and denominator.*

III. *Divide both numerator and denominator by this greatest common measure,* (by Rule under Art. 43,) *and the resulting fraction will be in its lowest terms.*

EXAMPLES.

1. Reduce $\dfrac{375a^3bxy}{15ab^2xy^3}$ to its lowest terms.

The greatest common measure of 375 and 15 is 15, to which annexing $abxy$, we have $15abxy$ for the greatest common measure of both numerator and denominator. Dividing the numerator by $15abxy$, we find

$$375a^3bxy \div 15abxy = 25a^2.$$

In the same way we find

$$15ab^2xy^3 \div 15abxy = by^2;$$

hence, we have

$$\frac{375a^3bxy}{15ab^2xy^3} = \frac{25a^2}{by^2};$$

which, by Rule under Art. 44, becomes

$$\frac{25a^2}{by^2} = 25a^2b^{-1}y^{-2}.$$

2. Reduce $\dfrac{42ax^3yz^5}{35xy^3z^3}$ to its lowest terms.

In this example, the greatest common measure of the numerator and denominator is $7xyz^3$; hence, dividing both numerator and denominator of our fraction by $7xyz^3$, we find

$\dfrac{42ax^3yz^5}{35xy^3z^3} = \dfrac{6ax^2z^2}{5y^2}$, which is in its lowest terms.

3. Reduce $\dfrac{-18mnx^2y^3}{72mx^3y^6}$ to its lowest terms.

Ans. $-\dfrac{n}{4xy^3}$.

4. What is the simplest form of $\dfrac{13x^3}{26xy^4}$?

Ans. $\dfrac{x^2}{2y^4}$.

5. What is the simplest form of $\frac{108ab^3cd^7}{12abcd}$?

Ans. $9b^2d^6$.

(49.) *From what has been said* (Art. 42), *we infer that we may transfer a letter from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of the exponent.*

Thus,

1. $\frac{7axyz}{ax^3y^6}=\frac{7z}{x^2y^5}=7zx^{-2}y^{-5}=\frac{7}{x^2y^5z^{-1}}.$

2. $17a^{-3}x^2z^{-1}=\frac{17x^2}{a^3z}=\frac{17}{a^3x^{-2}z}.$

3. Reduce $\frac{49abc^5}{35a^4b^2}$ to its simplest terms and then transfer all the letters to the numerator.

Ans. $\frac{7c^5}{5a^3b}=\frac{7}{5}a^{-3}b^{-1}c^5.$

4. Reduce in a similar manner the fraction $\frac{27abcd}{108a^3b^4m}$.

Ans. $\frac{cd}{4a^2b^3m}=\frac{1}{4}a^{-2}b^{-3}cdm^{-1}.$

GREATEST COMMON MEASURE OF POLYNOMIALS.

(50.) Before proceeding to the reduction of polynomial fractions, it is necessary to show how to find the greatest common measure of two polynomials, which may be effected by this

RULE.

Divide one of the polynomials by the other, and the preceding divisor by the last remainder, till nothing remains; the last divisor will be the greatest common measure.

This rule may be demonstrated as follows:

(51.) Let N and n be two polynomials, of which N is greater than n; then, performing the divisions as directed in the above rule, we have

OPERATION.

$n)\,N\,(q_1 =$ first quotient.

$\underline{nq_1}$

First remainder $= r_1)\ n\ (q_2 =$ second quotient.

$\underline{r_1q_2}$

Second remainder $= r_2)\ r_1\ (q_3 =$ third quotient.

$\underline{r_2q_3}$

Third remainder $= 0$

The numerals placed at the bottom of the letters q and r, are called *Subscript Numbers*, and show the order in which the quotients and remainders occur.

Letters marked like the above, are as independent as though they were different letters. The reason why we use them in preference to different letters, is because we can the more readily discover what they are designed to represent.

(52.) Now, since the dividend equals the divisor multiplied by the quotient and increased by the remainder, we have the following conditions:

$$N = q_1n + r_1. \qquad (1)$$
$$n = q_2r_1 + r_2. \qquad (2)$$
$$r_1 = q_3r_2. \qquad (3)$$

Substituting q_3r_2 for r_1 in (1) and (2), and they will become

$$N = q_1n + q_3r_2. \qquad (4)$$
$$n = q_2q_3r_2 + r_2. \qquad (5)$$

The right-hand member of (5) is divisible by r_2, and therefore its left-hand member must also be divisible by r_2; that is, n is divisible by r_2.

The value of n, (5), being substituted in (4), gives

$$\mathcal{N} = q_1q_2q_3r_2 + q_1r_2 + q_3r_2. \qquad (6)$$

The right-hand member of (6) will divide by r_2, and therefore its left-hand member will also divide by r_2; that is, $\mathcal{N}$ is divisible by r_2: hence, r_2 is a common measure of $\mathcal{N}$ and n. It is also the *greatest* common measure. For every common measure of $\mathcal{N}$ and n, is also a measure of $\mathcal{N} - nq_1 = r_1$; and every common measure of n and r_1, is also a measure of $n - r_1q_2 = r_2$. But the *greatest* measure of r_2 is *itself*. This, then, is the greatest common measure of $\mathcal{N}$ and n.

In the above case we have supposed the third remainder r_3 to $= 0$. Had the process of dividing extended still farther, it might still be shown, that the last divisor is the greatest common measure; hence the truth of the above rule.

(53.) It is obvious, that any factor common to but one of the two polynomials, may be struck out before dividing, without affecting the accuracy of the work.

(54.) Also, either of the polynomials may be multiplied by any factor before dividing.*

EXAMPLES.

1. What is the greatest common measure of $a^4 - x^4$, and $a^3 + a^2x - ax^2 - x^3$?

Arranging the terms according to the powers of a, and dividing according to Rule under Art. 46, we have for the

* If the above demonstration is deemed too difficult, on account of its making use of some of the principles of equations, which have not yet been fully explained, the student must pass it by, until he has gone through with the chapter on simple equations, and then he can return to it with pleasure and profit.

FIRST OPERATION.

$$\begin{array}{l|l} a^4 - x^4 & a^3 + a^2x - ax^2 - x^3 \\ \underline{a^4 + a^3x - a^2x^2 - ax^3} & \overline{\quad a - x \quad} \\ \quad - a^3x + a^2x^2 + ax^3 - x^4 & \\ \quad \underline{- a^3x - a^2x^2 + ax^3 + x^4} & \\ \qquad 2a^2x^2 - 2x^4 = \text{first remainder.} & \end{array}$$

We must now divide $a^3 + a^2x - ax^2 - x^3$ by $2a^2x^2 - 2x^4$; but before performing the division, we will expunge from $2a^2x^2 - 2x^4$ the factor $2x^2$ (Art. 53), which gives $a^2 - x^2$ for the divisor ; hence, we have for the

SECOND OPERATION.

$$\begin{array}{l|l} a^3 + a^2x - ax^2 - x^3 & a^2 - x^2 \\ \underline{a^3 \qquad\quad - ax^2} & \overline{a + x} \\ \qquad a^2x \qquad - x^3 & \\ \qquad \underline{a^2x \qquad - x^3} & \\ \qquad\qquad 0 & \end{array}$$

There being no remainder, the process must terminate. The last divisor, or greatest common measure, is therefore $a^2 - x^2$.

2. What is the greatest common measure of $6a^2 + 11ax + 3x^2$ and $6a^2 + 7ax - 3x^2$?

In this example, we may take either of the polynomials as the divisor, since they are each of the same degree.

FIRST OPERATION.

$$\begin{array}{l|l} 6a^2 + 11ax + 3x^2 & 6a^2 + 7ax - 3x^2 \\ \underline{6a^2 + 7ax - 3x^2} & \overline{\qquad 1 \qquad} \\ \qquad 4ax + 6x^2 = \text{first remainder.} & \end{array}$$

Before dividing $6a^2 + 7ax - 3x^2$ by $4ax + 6x^2$ we expunge from $4ax + 6x^2$ the factor $2x$, and thus have

SECOND OPERATION.

$$\begin{array}{r|l} 6a^2 + 7ax - 3x^2 & 2a + 3x \\ \cline{2-2} 6a^2 + 9ax & 3a - x \\ \hline -2ax - 3x^2 & \\ -2ax - 3x^2 & \\ \cline{1-1} 0 & \end{array}$$

Therefore, $2a + 3x$ is the greatest common measure.

3. What is the greatest common measure of

$$a^3 - a^2b + 3ab^2 - 3b^3 \text{ and } a^2 - 5ab + 4b^2\,?$$

FIRST OPERATION.

$$\begin{array}{r|l} a^3 - a^2b + 3ab^2 - 3b^3 & a^2 - 5ab + 4b^2 \\ \cline{2-2} a^3 - 5a^2b + 4ab^2 & a + 4b \\ \hline 4a^2b - ab^2 - 3b^3 & \\ 4a^2b - 20ab^2 + 16b^3 & \\ \cline{1-1} 19ab^2 - 19b^3 & \end{array}$$

$19ab^2 - 19b^3 =$ first remainder.

Before dividing $a^2 - 5ab + 4b^2$ by $19ab^2 - 19b^3$, we expunge from this last polynomial the factor $19b^2$.

SECOND OPERATION.

$$\begin{array}{r|l} a^2 - 5ab + 4b^2 & a - b \\ \cline{2-2} a^2 - ab & a - 4b \\ \hline -4ab + 4b^2 & \\ -4ab + 4b^2 & \\ \cline{1-1} 0 & \end{array}$$

Therefore, $a - b$ is the greatest common measure.

4. We will now seek the greatest common measure of these polynomials after the terms have been arranged according to the powers of b, as follows:

$$-3b^3 + 3ab^2 - a^2b + a^3 \text{ and } 4b^2 - 5ab + a^2\,?$$

Before dividing, we must multiply the polynomial $-3b^3 + 3ab^2 - a^2b + a^3$ by 4, in order that its left-hand term may be divisible by the left-hand term of the other polynomial. (Art. 54.)

FIRST OPERATION.

$$\begin{array}{r|l} -12b^3 + 12ab^2 - 4a^2b + 4a^3 & 4b^2 - 5ab + a^2 \\ \cline{2-2} -12b^3 + 15ab^2 - 3a^2b & -3b - 3a \\ \hline \text{Multiplying by 4,} \; -3ab^2 - a^2b + 4a^3 & \\ -12ab^2 - 4a^2b + 16a^3 & \\ -12ab^2 + 15a^2b - 3a^3 & \\ \hline -19a^2b + 19a^3 = \text{first remainder.} & \end{array}$$

Before dividing $4b^2 - 5ab + a^2$ by $-19a^2b + 19a^3$, we expunge from this last polynomial, the factor $19a^2$, and then dividing, we have for the

SECOND OPERATION.

$$\begin{array}{r|l} 4b^2 - 5ab + a^2 & -b + a \\ \cline{2-2} 4b^2 - 4ab & -4b + a \\ \hline -ab + a^2 & \\ -ab + a^2 & \\ \hline 0 & \end{array}$$

Therefore, $-b + a$, or $a - b$, is the greatest common measure, same as before.

4. What is the greatest common measure of the two polynomials $\begin{cases} 15a^5 + 10a^4b + 4a^3b^2 + 6a^2b^3 - 3ab^4, \\ 12a^3b^2 + 38a^2b^3 + 16ab^4 - 10b^5 ? \end{cases}$

Ans. $3a^2 + 2ab - b^2$.

5. What is the greatest common measure of

$$x^3 - b^2x \text{ and } x^2 + 2bx + b^2 ?$$

Ans. $x + b$.

6. What is the greatest common measure of

$$a^2 - ab - 2b^2 \text{ and } a^2 - 3ab + 2b^2\,?$$

Ans. $a - 2b$.

In this example it is immaterial which polynomial we consider as the divisor, since they are of the same degree.

7. What is the greatest common divisor of

$$\begin{cases} x^6 + 4x^5 - 3x^4 - 16x^3 + 11x^2 + 12x - 9, \\ 6x^5 + 20x^4 - 12x^3 - 48x^2 + 22x + 12\,? \end{cases}$$

Ans. $x^3 + x^2 - 5x + 3$.

8. What is the greatest common divisor of

$$\begin{cases} 20x^6 - 12x^5 + 16x^4 - 15x^3 + 14x^2 - 15x + 4, \\ 15x^4 - 9x^3 + 47x^2 - 21x + 28\,? \end{cases}$$

Ans. $5x^2 - 3x + 4$.

CASE II.

(55.) To reduce a polynomial fraction, that is, a fraction of which the numerator or denominator, or both, are polynomials, to its lowest terms, we have this

RULE.

Divide both numerator and denominator by their greatest common measure, found by Rule under Art. 50.

EXAMPLES.

1. Reduce the fraction $\dfrac{36x^6 - 18x^5 - 27x^4 + 9x^3}{27x^5y^2 - 18x^4y^2 - 9x^3y^2}$ to its simplest form.

We see, by a mere glance of the eye, that the numerator and denominator can both be divided by $9x^3$, by which division the fraction becomes $\dfrac{4x^3 - 2x^2 - 3x + 1}{3x^2y^2 - 2xy^2 - y^2}$.

We must now seek the greatest common measure of

$$4x^3 - 2x^2 - 3x + 1 \text{ and } 3x^2y^2 - 2xy^2 - y^2.$$

Dividing the second of these by y^2 (Art. 53), and multiplying the first by 3 (Art. 54), we have the

FIRST OPERATION.

$$\begin{array}{l|l}
12x^3 - 6x^2 - 9x + 3 & 3x^2 - 2x - 1 \\ \cline{2-2}
12x^3 - 8x^2 - 4x & 4x + 2 \\ \cline{1-1}
\end{array}$$

$$\begin{array}{rr}
\text{Multiplying by 3,} & 2x^2 - 5x + 3 \\
 & 6x^2 - 15x + 9 \\
 & 6x^2 - 4x - 2 \\ \cline{2-2}
 & -11x + 11 = \text{first remainder.}
\end{array}$$

We must now repeat the operation upon $3x^2 - 2x - 1$ and $-11x + 11$. Dividing the second of these by 11 (Art. 53), we have for the

SECOND OPERATION.

$$\begin{array}{r|l}
3x^2 - 2x - 1 & -x + 1 \\ \cline{2-2}
3x^2 - 3x \phantom{{}-1} & -3x - 1 \\ \cline{1-1}
x - 1 & \\
x - 1 & \\ \cline{1-1}
0 &
\end{array}$$

Hence, the greatest common measure of the numerator denominator of the fraction $\dfrac{4x^3 - 2x^2 - 3x + 1}{3x^2y^2 - 2xy^2 - y^2}$ is $-x + 1$ or $x - 1$. Dividing both numerator and denominator, of the above fraction, by $x - 1$, it becomes $\dfrac{4x^2 + 2x - 1}{3xy^2 + y^2}$ for the reduced value of the given fraction.

2\. Reduce $\dfrac{x^3 - xy^2}{x^2 + 2xy + y^2}$ to its lowest terms.

In this example the greatest common measure of the numerator and denominator is found to be $x+y$. Hence, the fraction reduced becomes $\frac{x^2-xy}{x+y}$.

3. Reduce $\frac{m^4-n^4}{m^3-m^2n-mn^2+n^3}$ to its simplest form.

Ans. $\frac{m^2+n^2}{m-n}$.

4. Reduce $\frac{a^2-ab-2b^2}{a^2-3ab-2b^2}$ to its simplest form.

Ans. $\frac{a+b}{a-b}$.

5. Reduce $\frac{6xy+8x+9y+12}{10xy-8x+15y-12}$ to its simplest form.

Ans. $\frac{3y+4}{5y-4}$.

6. Reduce $\frac{6x^5-4x^4-11x^3-3x^2-3x-1}{4x^4+2x^3-18x^2+3x-5}$ to its simplest form.

Ans. $\frac{3x^2+4x+1}{2x+5}$.

CASE III.

(56.) **To reduce a mixed quantity to the form of a fraction.**

RULE.

Multiply the entire part by the denominator of the fraction, to which product add the numerator, and under the result place the given denominator.

EXAMPLES.

1. Reduce $11x+\frac{x+y}{7x}$ to the form of a fraction.

In this example the entire part is $11x$, which multiplied by the denominator $7x$, gives $77x^2$, to which adding the numerator $x+y$, we have $77x^2+x+y$ for the numerator of the fraction sought, under which placing the denominator $7x$, we finally obtain $\frac{77x^2+x+y}{7x}$ for the reduced form of $11x+\frac{x+y}{7x}$.

2. Reduce $x-\frac{bx+x^2}{m}$ to the the form of a fraction.

Ans. $\frac{mx-bx-x^2}{m}$.

3. Reduce $y+3x-\frac{6}{3+a}$ to the form of a fraction.

Ans. $\frac{3y+9x+ay+3ax-6}{3+a}$.

4. Reduce $x-\frac{a^2-b^2}{x}$ to the form of a fraction.

Ans. $\frac{x^2-a^2+b^2}{x}$.

5. Reduce $3a^2-6+\frac{6-x^2}{7-y}$ to the form of a fraction.

Ans. $\frac{21a^2-36-3a^2y+6y-x^2}{7-y}$.

6. Reduce $9+\frac{3x^2-8c^4}{a-x^2}$ to the form of a fraction.

Ans. $\frac{9a-6x^2-8c^4}{a-x^2}$.

CASE IV.

(57.) To reduce a fraction to an entire or mixed quantity.

RULE.

Divide the numerator by the denominator, the quotient will be the entire part; if there is a remainder, place it over the denominator for the fractional part.

EXAMPLES.

1. Reduce $\frac{9a - 6x^2 - 8c^4}{a - x^2}$ to a mixed quantity.

Dividing the numerator by the denominator, we find this

FIRST OPERATION.

$$\begin{array}{l|l} 9a - 6x^2 - 8c^4 & a - x^2 \\ 9a - 9x^2 & \overline{9 = \text{integral part.}} \\ \hline \qquad 3x^2 - 8c^4 = \text{numerator of fractional part.} & \end{array}$$

Therefore the quantity sought is $9 + \frac{3x^2 - 8c^4}{a - x^2}$.

We will now change the order of the terms of the numerator and denominator, by placing the x^2 first; we thus find this

SECOND OPERATION.

$$\begin{array}{l|l} -6x^2 + 9a - 8c^4 & -x^2 + a \\ -6x^2 + 6a & \overline{\quad 6 = \text{integral part.}} \\ \hline \qquad 3a - 8c^4 = \text{numerator of fractional part} & \end{array}$$

Therefore the quantity sought is $6 + \frac{3a - 8c^4}{a - x^2}$.

These two results are equivalent, but under different forms.

2. Reduce $\frac{ax - x^2}{x}$ to an entire quantity.

Ans $a - x$.

3. Reduce $\frac{6x^2 - ax}{3x + 1}$ to a mixed quantity.

Ans. $2x - \frac{2x + ax}{3x + 1}$.

4. Reduce $\frac{m^3 - y^3}{m - y}$ to an entire quantity.

Ans. $m^2 + my + y^2$.

5. Reduce $\frac{20a^2 - 10a + 6}{5a}$ to a mixed quantity.

Ans. $4a - 2 + \frac{6}{5a}$.

6. Reduce $\frac{9y^3 - 18y + 8a^2y^2}{9y}$ to a mixed quantity.

Ans. $y^2 - 2 + \frac{8a^2y}{9}$.

7. Reduce $\frac{14m^3 - 21n}{7m}$ to a mixed quantity.

Ans. $2m^2 - \frac{3n}{m}$.

CASE V.

(58.) To reduce fractions to a common denominator.

RULE.

Multiply successively each numerator into all the denominators, except its own, for new numerators, and all the denominators together for a common denominator.

EXAMPLES.

1. Reduce $\frac{a}{x}$, $\frac{b}{2}$, $\frac{c}{7a}$ to equivalent fractions having a common denominator.

$a \times 2 \times 7a = 14a^2 =$ new numerator of first fraction.
$b \times x \times 7a = 7abx =$ new numerator of second fraction.
$c \times x \times 2 \ = 2cx =$ new numerator of third fraction.
and $x \times 2 \times 7a = 14ax =$ common denominator.

Therefore, $\frac{14a^2}{14ax}$; $\frac{7abx}{14ax}$; $\frac{2cx}{14ax}$ are the equivalent fractions sought.

2. Reduce $\frac{3m}{2a}$, $\frac{2b}{3x}$, and y, to fractions having a common denominator.

Ans. $\frac{9mx}{6ax}$; $\frac{4ab}{6ax}$; $\frac{6axy}{6ax}$.

3. Reduce $\frac{1}{2}$, $\frac{x^2}{3}$, $\frac{a^2+x^2}{a+x}$ to equivalent fractions having a common denominator.

Ans. $\frac{3a+3x}{6a+6x}$; $\frac{2ax^2+2x^3}{6a+6x}$; $\frac{6a^2+6x^2}{6a+6x}$.

4. Reduce $\frac{x}{3b}$, $\frac{6x^2}{5c}$, $\frac{a^2-x^2}{d}$ to fractions having a common denominator.

Ans. $\frac{5cdx}{15bcd}$; $\frac{18bdx^2}{15bcd}$; $\frac{15a^2bc-15bcx^2}{15bcd}$.

5. Reduce $\frac{x}{2}$, $\frac{x-1}{3}$, $\frac{x^2+2}{4}$ to fractions having a common denominator.

Ans. $\frac{12x}{24}$; $\frac{8x-8}{24}$; $\frac{6x^2+12}{24}$.

CASE VI.

(59.) To add fractional quantities.

RULE.

Reduce the fractions to a common denominator; then add the numerators, and place their sum over the common denominator.

EXAMPLES.

1. What is the sum of $\frac{x}{3a}$, $\frac{1}{3}$, $\frac{y}{7}$?

These fractions, when reduced to a common denominator, become $\frac{21x}{63a}$, $\frac{21a}{63a}$, $\frac{9ay}{63a}$; adding their numerators, we have $21x+21a+9ay$; placing this over the common denominator, we find

$$\frac{x}{3a}+\frac{1}{3}+\frac{y}{7}=\frac{21x+21a+9ay}{63a}=\frac{7x+7a+3ay}{21a}$$

2. What is the sum of $3x+\frac{2x}{5}$ and $x-\frac{8x}{9}$?

Ans. $3x+\frac{23x}{45}$.

3. What is the sum of $\frac{2x}{3}$, $\frac{7x}{4}$, $\frac{2x+1}{5}$?

Ans. $2x+\frac{49x+12}{60}$.

4. What is the sum of $\frac{3x}{4}$, $\frac{4x}{5}$, $\frac{5x}{6}$?

Ans. $\frac{45x+48x+50x}{60}=2x+\frac{23x}{60}$.

5. What is the sum of $\frac{a+b}{2}$, $\frac{a-b}{2}$?

Ans. a.

6. What is the sum of $\frac{a^2+2ab+b^2}{4}$, $\frac{a^2-2ab+b^2}{4}$?

Ans. $\frac{a^2+b^2}{2}$.

CASE VII.

(60.) To subtract one fraction from another.

RULE.

Reduce the fractions to a common denominator, then subtract the numerator of the subtrahend from the numerator of the minuend, and place the difference over the common denominator.

EXAMPLES.

1. From $\frac{3x+a}{4}$ subtract $\frac{2x-a}{3}$.

These fractions, when reduced to a common denominator, become $\frac{9x+3a}{12}$ and $\frac{8x-4a}{12}$. Subtracting the numerators we have $9x+3a-(8x-4a)=x+7a$, placing this over the common denominator 12, we find

$$\frac{3x+a}{4}-\frac{2x-a}{3}=\frac{x+7a}{12}.$$

2. From $\frac{6m+y}{5}$ subtract $\frac{m+y}{4}$.

Ans. $\frac{19m-y}{20}$.

3. From $3y+\frac{y}{a}$ subtract $y-\frac{y-a}{c}$.

Ans. $2y+\frac{cy+ay-a^2}{ac}$.

4. From $\frac{x+y}{2}$ subtract $\frac{x-y}{2}$.

Ans. y.

5. From $\frac{x^2+2xy+y^2}{4xy}$ subtract $\frac{x^2-2xy+y^2}{4xy}$.

Ans. 1.

6. From $\frac{6-x}{2}$ subtract $a+\frac{3+y}{3}$.

Ans. $2-a-\frac{3x+2y}{6}$.

CASE VIII.

(61.) To multiply fractional quantities together.

RULE.

If any of the quantities to be multiplied are mixed, they must, by Case III, be reduced to a fractional form; then multiply together all the numerators for a numerator, and all the denominators together for a denominator.

EXAMPLES.

1. Multiply $\frac{x+a}{2}$ by $\frac{x+b}{3}$.

The product of the numerators will be

$$(x+a)\times(x+b)=x^2+ax+bx+ab;$$

and the product of the denominators is $2\times3=6$.

Hence, $\frac{x+a}{2}\times\frac{x+b}{3}=\frac{x^2+ax+bx+ab}{6}$.

2. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$.

Ans. $\frac{x^4-b^4}{b^2c+bc^2}$.

3. What is the continued product of $\frac{3-x}{7}$, $\frac{4+x}{2}$, and $\frac{3}{7}$?

Ans. $\frac{36-3x-3x^2}{98}$.

4. What is the product of $\frac{a+b}{2}$, $\frac{a-b}{2}$?

Ans. $\frac{a^2-b^2}{4}$.

5. What is the continued product of $\frac{2x}{m}$, $\frac{h-d}{x}$, $\frac{b}{c}$, and $\frac{1}{r-1}$?

Ans. $\frac{2bhx-2bdx}{cmrx-cmx}$.

6. What is the product of $y+\frac{y+1}{b}$ by $\frac{y-1}{2b}$?

Ans. $\frac{by^2-by+y^2-1}{2b^2}$.

CASE IX.

(62.) To divide one fraction by another.

RULE.

If there are any mixed quantities, reduce them to a fractional form, by Case III.; then invert the divisor, and multiply as in Case VIII.

EXAMPLES.

1. Divide $\frac{3x+7}{4}$ by $\frac{4x-1}{5}$.

If we invert the divisor, and then multiply, we have

$\frac{3x+7}{4}\times\frac{5}{4x-1}=\frac{15x+35}{16x-4}$ for the quotient.

2. Divide $\frac{x^2-y^2}{x}$ by $\frac{x^2+y^2}{y}$.

Ans. $\frac{x^2y-y^3}{x^3+xy^2}$.

3. Divide $\frac{7a^2y^{-3}}{3m^3}$ by $\frac{4ay^2}{5m}$.

Ans. $\frac{35a}{12m^2y^5}$.

4. Divide $\frac{y-b}{8cd}$ by $\frac{3cy}{4d}$.

Ans. $\frac{y-b}{6c^2y}$.

5. What is the quotient of $\frac{x}{x-1}$ divided by $\frac{x}{2}$?

Ans. $\frac{2}{x-1}$.

6. What is the quotient of $\frac{x^2-3}{7}$ divded by $\frac{x-1}{7}$?

Ans. $\frac{x^2-3}{x-1}$.

CHAPTER III.

SIMPLE EQUATIONS.

(63.) *An equation* is an expression of two equal quantities with the sign of equality placed between them.

The terms or quantities on the left-hand side of the sign of equality constitute the *first member* of the equation, those on the right constitute the *second member*.

Thus,
$$x + 2 = a, \qquad (1)$$
$$\frac{x}{2} - 1 = b, \qquad (2)$$
$$3x + 7 = c, \qquad (3)$$
are equations; the first is read, "x increased by 2 equals a."

The second is read, "one-half of x diminished by 1 equals b."

The third is read, "three times x increased by 7 equals c."

(64.) Nearly all the operations of algebra are carried on by the aid of equations. The relations of a question or problem are first to be expressed by an equation, containing known quantities as well as the unknown quantity. Afterwards we must make such transformations upon this equation as to bring the unknown quantity by itself on one side of the equation, by which means it becomes known.

(65.) *An equation of the first degree, or a simple equation*, is one, in which the unknown has no power above the first degree.

(66.) *A quadratic equation*, is an equation of the second degree, that is, the unknown quantity is involved to the second power, and to no greater power.

(67.) An equation of the third, fourth, &c., degree, is one which contains the unknown quantity to the third, fourth, &c., degree ; but to no superior degree.

And in general, an equation which involves the mth power of the unknown quantity, is called an equation of the mth degree.

(68.) The following axioms will enable us to make many transformations upon the terms of an equation without destroying their equality.

AXIOMS.

I. *If equal quantities be added to both members of an equation, the equality of the members will not be destroyed.*

II. *If equal quantities be subtracted from both members of an equation, the equality of the members will not be destroyed.*

III. *If both members of an equation be multiplied by the same quantity, the equality will not be destroyed.*

IV. *If both members of an equation be divided by the same quantity, the equality will not be destroyed.*

CLEARING EQUATIONS OF FRACTIONS.

(69.) When some of the terms of an equation are fractional, it is necessary to so transform it, as to cause the denominators to disappear, which process is called *clearing of fractions*.

Let it be required to clear of fractions, the following equation.

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{6}=x+1. \qquad (1)$$

Now, by Axiom III, we can multiply all the terms of this equation by any number we please, without destroying the equality. If we multiply by a multiple of all the denominators, it is evident they will disappear.

If we choose the least multiple of the denominators as a multiplier, it is plain that the labor of multiplying will be the least possible.

Thus, in the above example, multiplying all the terms of both sides of the equation by 6, which is the least multiple of 2, 3, and 6, we have

$$3x+2x+x=6x+6. \qquad (2)$$

This equation is now free of fractions.

(70.) Hence, to clear an equation of fractions, we deduce, from what has been said, this

RULE.

Multiply all the terms of the equation by any multiple of their denominators. If we choose the least common multiple of the denominators, for our multiplier, the terms of the fraction, when cleared, will be in their simplest form.

EXAMPLES.

1. Clear of fractions the equation $\frac{x-a}{5}=\frac{x+b}{2}-\frac{1}{7}$.

In this example the least common multiple of the denominators 5, 2, and 7, is 70; hence, multiplying all the terms of our equation by 70, we find

$$14x-14a=35x+35b-10,$$

for the equation when cleared of fractions.

2. Clear of fractions $\frac{x-2}{8}+\frac{x+a}{4}-\frac{x-b}{2}=x+\frac{x}{16}$.

Ans. $2x-4+4x+4a-8x+8b=16x+x$.

(71.) We must observe that when a fraction has the sign —, it requires its value to be subtracted, so that, if it is written without the denominator, all the signs of the numerator must be changed.

3. Clear of fractions $\frac{x-1}{2}+\frac{x+1}{3}-\frac{x-3}{4}=a+b-\frac{c}{7}$.

Ans. $42x-42+28x+28-21x+63=84a+84b-12c$.

4. Clear the equation $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}+\frac{x}{5}+\frac{x}{6}=251$ of fractions.

Ans. $30x+20x+15x+12x+10x=15060$.

5. Clear the equation $\frac{a^2}{x}+\frac{b}{m}+\frac{c}{d}=g$ of fractions.

Ans. $a^2dm+bdx+cmx=dgmx$.

6. Clear the equation $\frac{x}{a^2-b^2}+\frac{x-3}{a+b}-\frac{x-5}{a-b}=\frac{m}{a}$ of fractions.

Ans. $\left\{\begin{array}{l} ax+a^2x-abx-3a^2+3ab-a^2x-abx+5a^2 \\ +5ab=a^2m-b^2m. \end{array}\right.$

TRANSPOSITION OF THE TERMS OF AN EQUATION.

(72.) The next thing to be attended to, after clearing the equation of fractions, is to transform it so that all the terms containing the unknown quantity may constitute one member of the equation.

If we take the equation

$$\frac{a}{x}-\frac{1}{2}+\frac{b}{3}=8, \qquad (1)$$

we have, when cleared of fractions,

$$6a - 3x + 2bx = 48x. \qquad (2)$$

If we add to both members of this equation $3x - 2bx$ (Axiom I.), it becomes

$$6a - 3x + 2bx + 3x - 2bx = 48x + 3x - 2bx. \qquad (3)$$

All the terms of the left-hand member cancel each other, except $6a$.

Therefore we have

$$6a = 48x + 3x - 2bx, \qquad (4)$$

in which all the terms of the right-hand member contain x.

If we compare equation (4) with (2), we shall discover, that the terms $-3x + 2bx$, which are on the left side of equation (2), are on the right side of equation (4), with their signs changed.

Hence, we conclude that the terms of an equation may change sides, provided they change signs at the same time.

(73.) To transpose a term from one side of an equation to the other, we must observe this

RULE.

Any term may be transposed from one side of an equation to the other, by changing its sign.

EXAMPLES.

1. Clear the equation $\frac{x+6}{2} + 26 = \frac{5x}{4} + 2$ of fractions, and transpose the terms so that all those containing x may constitute the left-hand member.

First, clearing the above equation of fractions, by Rule under Art. 70, we have

$$2x + 12 + 104 = 5x + 8.$$

Secondly, transposing $2x$ from the left member to the right member, and 8 from the right member to the left, we have $12+104-8=5x-2x$ for the result required.

2. Clear the equation $\frac{x}{2}-\frac{a+x}{3}=7\frac{1}{2}$ of fractions, and transpose the terms.

Ans. $3x-2x=45+2a$.

3. Clear of fractions, the equation $\frac{7x}{9}-3\frac{1}{3}+\frac{x}{6}=\frac{x}{9}+2$ and transpose the terms.

Ans. $14x+3x-2x=36+60$.

4. Clear of fractions and transpose the terms of the equation $\frac{x}{a-b}-\frac{2+x}{a+b}=\frac{c}{a^2-b^2}$.

Ans. $ax+bx-ax+bx=c+2a-2b$.

(74.) We are now prepared to find the value of the unknown quantity. If we take the last example, it may be written thus,

$$(a+b-a+b)\,x=c+2a-2b;$$

or uniting the like terms within the parenthesis, it becomes

$$2bx=c+2a-2b.$$

Dividing both sides of this equation by $2b$, (Axiom IV.), we find
$$x=\frac{c+2a-2b}{2b};$$

hence, the value of x is now known, since it is equal to the expression $\frac{c+2a-2b}{2b}$.

(75.) From what has been done, we discover that an equation of the first degree may be resolved by the following general

RULE.

I. *If any of the terms of the equation are fractional, the equation must be cleared of fractions, by Rule under Art.* 70.

II. *The terms must then be so transposed, that all those containing the unknown quantity may constitute one side or member of the equation, by Rule under Art.* 73.

III. *Then divide the algebraic sum of those terms on that side of the equation which are independent of the unknown quantity, by the algebraic sum of the coefficients of the terms containing the unknown quantity, the quotient will be the value of the unknown quantity.*

EXAMPLES.

1. What is the value of x in the equation $\frac{x}{3}+\frac{x}{4}=x-10$?

This, cleared of fractions, becomes

$$4x+3x=12x-120.$$

When the terms are transposed and united, we have

$$120=5x.$$

Dividing by 5, we get $\quad 24=x.$

2. What is the value of x in the equation

$$x-\frac{2x+1}{3}=\frac{x+3}{4}?$$

Ans. $x=13.$

3. Given $\frac{21-3x}{3}-\frac{4x+6}{9}=6-\frac{5x+1}{4}$ to find x.

Ans. $x=3.$

4. Find x from the equation $3ax+\frac{a}{2}-3=bx-a$.

Ans. $x=\frac{6-3a}{6a-2b}.$

5. Given $\frac{x-2}{4} - \frac{3x}{2} + \frac{15x}{2} = 37$, to find x.

Ans. $x = 6$.

6. Find x so as to satisfy the condition $\frac{3cx}{a} - \frac{2bx}{m} - 4 = f$.

Ans. $x = \frac{afm + 4am}{3cm - 2ab}$.

7. Find x from the equation $\frac{8nx - b}{7} - \frac{3b}{2} = 4 - b - \frac{c}{2}$.

Ans. $x = \frac{56 + 9b - 7c}{16n}$.

8. Given $\frac{x-1}{2} + \frac{x+1}{2} = 3x - 12$, to find x.

Ans. $x = 6$.

9. Given $x - \frac{3x-5}{13} + \frac{4x-2}{11} = x + 1$, to find x.

Ans. $x = 6$.

10. Given $\frac{x+7}{3} - 3x + \frac{6x-2}{5} + 3 = x$, to find x.

Ans. $x = 2$.

11. Given $\frac{3x-2}{7} + \frac{3x+2}{11} = x - 1$, to find x.

Ans. $x = 3$.

12. Given $\frac{x}{3} - \frac{x}{7} + x = 11$, to find x.

Ans. $x = 9\frac{6}{25}$.

13. Given $\frac{(a+b)x}{a-b} + \frac{x}{a^2 - b^2} = \frac{x+1}{a+b}$, to find x.

Ans. $x = \frac{a-b}{a^2 + 2ab + b^2 - a + b + 1}$.

QUESTIONS, THE SOLUTION OF WHICH REQUIRE EQUATIONS OF THE FIRST DEGREE.

(76.) In the solution of questions, by the aid of algebra, the most difficult part is to obtain the proper equation which shall include all the necessary relations of the question. When once this equation of condition is properly found, the value of the unknown quantity is readily obtained by the Rule under Art. 75.

Suppose we wish to solve, by algebra, the following question.

1. What number is that, whose half increased by its third part and one more shall equal itself?

If we suppose x to be the number sought, its half will be $\frac{x}{2}$, which increased by its third part, becomes $\frac{x}{2}+\frac{x}{3}$, and this increased by one, becomes $\frac{x}{2}+\frac{x}{3}+1$, which by the question must equal itself.

Therefore, we have $\frac{x}{2}+\frac{x}{3}+1=x$ for the equation of condition.

Solving this, by Rule under Art. 75, we have $x=6$.

VERIFICATION.

$$\begin{aligned}\frac{x}{2} &= \tfrac{1}{2} \text{ of } 6 = 3,\\ \frac{x}{3} &= \tfrac{1}{3} \text{ of } 6 = 2,\\ 1 &= 1,\end{aligned}$$

Therefore, $\frac{x}{2}+\frac{x}{3}+1=6$, which shows that 6 is truly the number sought.

Again, let us endeavor to solve this question:

2. What number is that whose third part exceeds its fourth part by 5?

Suppose x to be the number, then will its third part $\frac{x}{3}$; its fourth part $=\frac{x}{4}$.

Therefore, the excess of its third part over its fourth part is expressed by $\frac{x}{3}-\frac{x}{4}$, which, by the question, must equal 5.

Hence, we have the following equation $\frac{x}{3}-\frac{x}{4}=5$,

this solved, gives $x=60$; the third part of which is 20, and its fourth part is 15, so that its third part exceeds its fourth part by 5, hence, this is the correct number sought.

(77.) The method of forming an equation from the conditions of a question, is of such a nature as not to admit of any simple rule, but must be in a measure left to the ingenuity of the student.

It will however be of assistance to pay attention to the following

RULE.

Having denoted the quantity sought by x, or some other letter, we must indicate, by algebraic symbols, the same operation, as it would be necessary to perform upon the true number, in order to verify the conditions of the question.

3. Out of a cask of wine which had leaked away a third part, 21 gallons were afterwards drawn, and the cask was then found to be half full: how much did it hold?

Suppose x to be the number of gallons which the cask held.

Then, the part leaked away must be $\frac{x}{3}$.

And the part leaked away, together with the quantity drawn off, must be $\frac{x}{3}+21$.

Now, by the question, the cask is still half full; so that what has leaked out, together with what has been drawn off must be $\frac{x}{2}$.

Hence, we have this equation, $\frac{x}{2}=\frac{x}{3}+21$, which, cleared of fractions, becomes, $3x=2x+126$; transposing and uniting terms, we have $x=126$.

4. There are two numbers which are to each other as 6 to 5, and whose difference is 40. What are the numbers.

Suppose the numbers to be denoted by $6x$ and $5x$, which are obviously as 6 to 5 for all values of x. Now, by the question, the difference of these numbers is 40. Therefore, we have $6x-5x=40$; that is, $x=40$.

Hence, $\left.\begin{array}{l}6x=6\times40=240\\5x=5\times40=200\end{array}\right\}$ the numbers sought.

5. A farmer had two flocks of sheep, each containing the same number. Having sold from one of these 39, and from the other 93, he finds twice as many remaining in the one as in the other. How many did each flock originally contain?

Suppose the number in each flock to be denoted by x.

Then the flock from which he sold 39 will have remaining $x-39$.

And the one from which he sold 93 will have remaining $x-93$.

Hence, by the the question, we have

$$2\times(x-93)=x-39, \text{ or } 2x-186=x-39;$$

transposing and uniting terms, $x=147$.

6. Divide the number 36 into three such parts, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, shall be equal to each other.

If we denote the three parts by $2x, 3x, 4x$, it is plain that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, $\frac{1}{4}$ of the third, will be equal for all values of x.

Now, by the question, the sum of these three parts must equal 36.

Therefore, $\quad 2x+3x+4x=36$;
uniting terms, we have $\quad 9x=36$;
dividing by 9, and we obtain $\quad x=4$.

Consequently, $\left.\begin{aligned} 2x &= 2\times4 = 8 \\ 3x &= 3\times4 = 12 \\ 4x &= 4\times4 = 16 \end{aligned}\right\}$ the parts sought.

7. Two pieces of cloth are of the same price by the yard, but of different lengths; the one cost \$5, the other \$6$\frac{1}{2}$. If each piece had been 10 yards longer, their lengths would have been as 5 to 6. What was the length of each piece?

Since the price per yard was the same for both pieces, their lengths must have been to each other the same as the number of dollars which they cost, or as 5 to $6\frac{1}{2}$, or, which is the same, as 10 to 13.

Therefore we will denote their lengths by $10x$ and $13x$.

These become, when increased by 10,

$$10x+10 \text{ and } 13x+10,$$

which, by the question, must be as 5 to 6.

Hence, $\quad 6(10x+10)=5(13x+10)$;
or, expanding, $\quad 60x+60=65x+50$;
transposing and uniting terms, we get $10=5x$, and $x=2$.

Therefore, $\left.\begin{aligned} 10x &= 10\times2 = 20 \\ 13x &= 13\times2 = 26 \end{aligned}\right\}$ the lengths sought.

8. Twelve oxen have in 4 weeks eaten all the grass which grew on $3\frac{1}{3}$ acres of land, in such a manner that they not only ate all the grass which at first was there, but also that which grew during the time they were grazing. In like manner, have 21 oxen, in 9 weeks, eaten all the grass upon 10 acres of land. How many oxen can, in this way, graze for 18 weeks upon 24 acres of land?

Let $x =$ the growth in acres of one acre of grass for one week; then will the growth of $3\frac{1}{3}$ acres for 4 weeks equal

$$3\tfrac{1}{3}\times4\times x=\frac{40x}{3};$$

also, the growth of 10 acres for 9 weeks will equal

$$10\times9\times x=90x.$$

Therefore, the whole quantity of grass eaten in the first case, equals

$$3\tfrac{1}{3}+\frac{40x}{3}=\frac{10+40x}{3}.$$

The quantity eaten in the second case equals $10+90x$. Hence the quantity which one ox eat in one week equals

$$\frac{10+40x}{3}\times\frac{1}{4}\times\frac{1}{12}=\frac{5+20x}{72}, \text{ in the first case.}$$

Again, in the second condition the quantity which one ox eat in one week equals

$$(10+90x)\times\frac{1}{9}\times\frac{1}{21}=\frac{10+90x}{189}.$$

Now, by the question, an ox in the first case ate the same as an ox in the second case; therefore we have

$$\frac{5+20x}{72}=\frac{10+90x}{189}. \qquad (1)$$

This, solved, gives $x=\frac{1}{12}$ of an acre.

This value of x substituted in either member of (1) gives

$\frac{5}{54}$ for the fractional part of an acre eaten by one ox in one week; therefore, the quantity which 1 ox eats in 18 weeks is

$$\frac{5}{54} \times 18 = \frac{5}{3} \text{ acres.}$$

Now, the 24 acres increasing 18 weeks, at the rate of $\frac{1}{12}$ of an acre for each acre for each week, will amount to 60 acres.

Hence, $\quad 60 \div \frac{5}{3} = 36$ oxen for the answer.

9. Divide the number 237 into two such parts, that the one may be contained in the other $1\frac{1}{4}$ times. What are these parts?

Ans. $105\frac{1}{3}$ and $131\frac{2}{3}$.

10. The sum of \$1200 is to be divided between two persons, A and B, so that A's share may be to B's share as 2 to 7. How much does each receive?

Ans. A \$$266\frac{2}{3}$, B \$$933\frac{1}{3}$.

The above question, when generalized, becomes like the following question.

11. Divide a number a into two such parts, that the first part may be to the second as m to n. What are the parts?

Ans. $\frac{ma}{m+n}, \frac{na}{m+n}$.

12. Divide the number 46 into two unequal parts, so that when the greater is divided by 7, and the less by 3, the quotients together may amount to 10. What are these parts?

Ans. 28 and 18.

13. In a company of 266 persons, consisting of officers, merchants, and students, there were four times as many merchants, and twice as many officers as students. How many were there of each class?

Ans. 38 students, 152 merchants, and 76 officers.

14. Divide the number a into three such parts, that the second may be m times, and the third n times as great as the first. What are these parts?

$$\text{Ans. } \frac{a}{1+m+n}, \quad \frac{ma}{1+m+n}, \quad \frac{na}{1+m+n}.$$

15. A field of 864 square rods is to be divided among three farmers, A, B, C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together. How much does each receive?

Ans. A 135, B 297, C 432 square rods.

16. Divide the number a into three such parts, that the first shall be to the second as m to n, and the second part to the third as p to q. What are these parts?

$$\text{Ans. } \frac{mpa}{mp+np+nq}, \quad \frac{npa}{mp+np+nq}, \quad \frac{nqa}{mp+np+nq}.$$

17. Divide \$1520 among three persons, A, B, C, so that B may receive \$100 more than A; and C \$270 more than B. How much does each receive?

Ans. A \$350, B \$450, C \$720.

18. A certain sum of money is to be divided amongst three persons, A, B, C, as follows: A shall receive \$3000 less than the half of it, B \$1000 less than the third part, and C is to receive \$800 *more* than the fourth part of the whole sum. What is the sum to be divided? and what does each receive?

Ans. The whole sum is \$38400. A receives \$16200, B \$11800, C \$10400.

19. A mason, 12 journeymen, and 4 assistants, receive together \$72 wages for a certain time. The mason receives \$1 daily, each journeyman $\frac{1}{2}$ dollar, and each assistant 25

cents. How many days must they have worked for this money ?

Ans. 9 days.

20. A courier who had started from a certain place 10 days ago, is pursued by another from the same place, and by the same way. The first goes 4 miles every day, the other 9. How many days will the second need to overtake the first ?

Ans. 8 days.

21. A courier left this place n days ago, and makes a miles daily. He is pursued by another making b miles daily. How many days will the second require to overtake the first ?

Ans. $\frac{na}{b-a}$ days.

22. But, in what time will the second courier overtake the first, when it is supposed that the second starts 12 days later than the first, and his speed is to that of the first as 8 is to 3 ?

Ans. $7\frac{1}{5}$ days.

23. Two bodies start from the same place, one after the other, in a straight line ; the second starts n seconds later than the first, and its speed is to that of the first as q is to p. In what time will these two bodies be together ?

Ans. $\frac{pn}{q-p}$ seconds after the setting out of the second.

24. Two bodies move in opposite directions ; one runs c feet in each second, the other C feet. The two places from which they start at the same time, are distant d feet from one another. When will they meet ?

Ans. $\frac{d}{C+c}$ seconds.

25. But, in what time will these two bodies come together, when that which goes C feet each second, runs after the other ?

Ans. $\frac{d}{C-c}$ seconds.

Is the problem, as here stated, always possible ? And what is required for it to be possible ? What does the expression $\frac{d}{C-c}$ signify, when $C=c$? What does it denote when $C<c$?

26. At 12 o'clock, both hands of a clock are together. When, and how often, will these hands be together in the next 12 hours ?

Ans. The hands will meet 11 times ; these rencounters will be at $5\frac{5}{11}$ minutes past 1, $10\frac{10}{11}$ minutes after 2, $16\frac{4}{11}$ after 3, and so on, in each successive hour $5\frac{5}{11}$ minutes later.

27. Two bodies move after one another in the circumference of a circle which measures p feet. At first they are distant from each other by an arc measuring d feet ; the first moves c feet, the second C feet in a second. When will these two bodies be together for the first time, second time, and so on, supposing that they do not disturb each other's motion ?

Ans. $\frac{d}{C-c}$, $\frac{p+d}{C-c}$, $\frac{2p+a}{C-c}$, &c., seconds.

28. But when will they meet, when the first begins to move t seconds sooner than the second ?

Ans. $\frac{d+ct}{C-c}$, $\frac{p+d+ct}{C-c}$, $\frac{2p+d+ct}{C-c}$, &c., seconds.

29. But if the first starts t seconds later than the second, when will they meet ?

Ans. $\frac{d-ct}{C-c}$, $\frac{p+d-ct}{C-c}$, $\frac{2p+d-ct}{C-c}$, &c., seconds.

30. But if the first, instead of preceding the second, runs against it, and starts from the same place t seconds sooner, when do they meet?

Ans. $\frac{d-ct}{C+c}$, $\frac{p+d-ct}{C+c}$, $\frac{2p+d-ct}{C+c}$, &c., seconds.

31. A cistern can be filled by three pipes; by the first in $1\frac{1}{3}$ hours, by the second in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled when all three pipes are open at once?

Ans. 48 minutes.

32. In order to make the foregoing problem more general, let the time which the first pipe alone takes in filling the cistern $=a$, the time which the second takes in doing the same $=b$, and the time required by the third $=c$. What expression gives the time in which all three pipes together will fill it?

Ans. $\frac{abc}{ab+ac+bc}$.

33. A servant received from his master \$40 wages, yearly, and a suit of livery. After he had served 5 months he asked for his discharge, and received for this time, the livery, together with $\$6\frac{1}{6}$ in money. How much did the livery cost?

Ans. \$18.

34. A master hired a journeyman, and promised him 8 shillings for each day that he worked for him; but if he worked anywhere else, then the journeyman must pay him 5 shillings daily for his board. At the expiration of 50 days they settle, and the journeyman receives 10 pounds and 18 shillings. How many days has he worked for his master?

Ans. 36 days.

35. Find a number such that $\frac{1}{3}$ thereof increased by $\frac{1}{4}$ of the same, shall be equal to $\frac{1}{6}$ of it if increased by 35.

Ans. 84.

36. A gentleman spends $\frac{2}{3}$ of his yearly income in board and lodging, $\frac{2}{3}$ of the remainder in clothes, and lays by \$200 a year. What is his income?

Ans. \$1800.

EQUATIONS OF TWO OR MORE UNKNOWN QUANTITIES.

(78.) Suppose we have given the two equations

$$x + y = 19,$$
$$x - y = 11,$$

to find the value of x and y.

If we take the sum of the two equations, we shall have

$$2x = 30.$$

Dividing by 2, we find

$$x = 15.$$

Again, subtracting the second equation from the first, we have

$$2y = 8.$$

Dividing by 2, we obtain

$$y = 4.$$

2. Suppose we nave given the two equations

$$\frac{x}{3} + \frac{y}{4} = 8, \qquad (1)$$

$$\frac{x}{6} + \frac{y}{16} = 3, \qquad (2)$$

to find the value of x and y.

We will first clear these equations of fractions, by multiplying the first by 12 and the second by 48 (Art. 70); we thus obtain

$$4x + 3y = 96, \qquad (3)$$
$$8x + 3y = 144. \qquad (4)$$

Now, subtracting (3) from (4) we have

$$4x = 48.$$

Divided by 4 we find

$$x = 12.$$

If we multiply (3) by 2 it becomes

$$8x + 6y = 192. \qquad (5)$$

Now, subtracting (4) from (5) we find

$$3y = 48.$$

Dividing by 3 we find

$$y = 16.$$

3. If we have given the two equations

$$2x - 3y = 4, \qquad (1)$$
$$8x - 6y = 40, \qquad (2)$$

to find x and y, we proceed as follows :

Dividing (2) by 2 it becomes

$$4x - 3y = 20. \qquad (3)$$

Subtracting (1 from (3) we find

$$2x = 16 \therefore x = 8.$$

Multiplying (1) by 4 we have

$$8x - 12y = 16. \qquad (4)$$

Subtracting (4) from (2) we get

$$6y = 24 \therefore y = 4.$$

ELIMINATION BY ADDITION AND SUBTRACTION.

(79.) From what has been done, we discover that an unknown quantity may be eliminated from two equations, by the following

RULE.

Operate upon the two given equations, by multiplication or division, so that the coefficients of the quantity to be eliminated may become the same in both equations; then add or subtract the two equations, as may be necessary, to cause these two terms to disappear.

EXAMPLES.

4. Given, to find x and y, the two equations

$$3x - y = 3, \qquad (1)$$
$$y + 2x = 7. \qquad (2)$$

If we add the two equations, we have

$$5x = 10 \therefore x = 2. \qquad (3)$$

Again, multiplying(3) by 2, we get

$$2x = 4. \qquad (4)$$

Subtracting (4) from (2) we obtain

$$y = 3.$$

5. Given, to find x and y, the two equations

$$\frac{x}{2} + \frac{y}{3} = 6, \qquad (1)$$
$$\frac{x}{3} + \frac{y}{2} = 6\tfrac{1}{2}. \qquad (2)$$

Clearing these equations of fractions, by multiplying each by 6, they become

$$3x+2y=36, \qquad (3)$$
$$2x+3y=39. \qquad (4)$$

Multiplying (3) by 3, and (4) by 2, they become

$$9x+6y=108, \qquad (5)$$
$$4x+6y=\ \ 78. \qquad (6)$$

Subtracting (6) from (5) we get

$$5x=30,$$
and $$x=6. \qquad (7)$$

Multiplying (7) by 2, it becomes

$$2x=12. \qquad (8)$$

Subtracting (8) from (4), we find

$$3y=27 \therefore y=9.$$

6. Suppose we wish to find x, y, and z, from the three equations

$$5x-6y+4z=15, \qquad (1)$$
$$7x+4y-3z=19, \qquad (2)$$
$$2x+\ \ y+6z=46. \qquad (3)$$

We will first eliminate y: for this purpose multiply (3), first by 4 and then by 6, and it will give

$$8x+4y+24z=184, \qquad (4)$$
$$12x+6y+36z=276. \qquad (5)$$

Add (1) to (5); and subtract (2) from (4), and we have

$$17x+40z=291, \qquad (6)$$
$$x+27z=165. \qquad (7)$$

We have now the two equations (6) and (7), and but two unknown quantities x and z.

Multiply (7) by 17 and it will become

$$17x+459z=2805. \qquad (8)$$

Subtracting (6) from (8) we obtain

$$419z = 2514. \qquad (9)$$

Dividing (9) by 419, we find

$$z = 6. \qquad (10)$$

Multiplying (10) by 27, we find

$$27z = 162. \qquad (11)$$

Subtracting (11) from (7) we get

$$x = 3. \qquad (12)$$

Multiplying (10) by 6, and (12) by 2, and then taking their sum, we find

$$6z + 2x = 42. \qquad (13)$$

Subtracting (13) from (3), we get

$$y = 4.$$

(80.) We will now repeat the solution of this last question, adopting a simple and easy method of indicating the successive steps in the operations.

The method which we propose to make use of, is to indicate by algebraic signs, the same operations upon the respective *numbers* of the different equations, as we wish to have performed upon the equations themselves.

Thus,

$(6) = (4) \times 3$ $\left\{\begin{array}{l}\text{shows, that equation (6) is obtained by} \\ \text{multiplying equation (4) by 3.}\end{array}\right.$

$(10) = (7) + (1)$ $\left\{\begin{array}{l}\text{shows, that equation (10) is obtained} \\ \text{by adding equations (7) and (1).}\end{array}\right.$

$(11) = (6) - (3)$ $\left\{\begin{array}{l}\text{shows, that equation (11) is obtained} \\ \text{by subtracting equation (3) from (6).}\end{array}\right.$

$(15) = (14) \div 3$ $\left\{\begin{array}{l}\text{shows, that equation (15) is obtained} \\ \text{by dividing equation (14) by 3.}\end{array}\right.$

And so on for other combinations.

This kind of notation will become familiar by a little practice.

We will now resume our equations of last example.

$$\text{Given}\left\{\begin{array}{ll} 5x-6y+4z=15 & (1)\\ 7x+4y-3z=19 & (2)\\ 2x+\ y+6z=46 & (3)\end{array}\right\}\text{, to find } x, y, \text{ and } z.$$

$$\begin{array}{rl}
8x+\ 4y+\ 24z=184 & (4)=(3)\times 4\\
12x+\ 6y+\ 36z=276 & (5)=(3)\times 6\\
17x+\ 40z=291 & (6)=(1)+(5)\\
x+27z\ =165 & (7)=(4)-(2)\\
17x+459z=2805 & (8)=(7)\times 17\\
419z=2514 & (9)=(8)-(6)\\
z=6 & (10)=(9)\div 419\\
27z=162 & (11)=(10)\times 27\\
x=3 & (12)=(7)-(11)\\
6z=36 & (13)=(10)\times 6\\
2x=6 & (14)=(12)\times 2\\
6z+2x=42 & (15)=(13)+(14)\\
y=4 & (16)=(3)-(15)
\end{array}$$

Collecting equations (12), (16), and (10), we have

$$\text{Ans.}\left\{\begin{array}{ll} x=3. & (12)\\ y=4. & (16)\\ z=6. & (10)\end{array}\right.$$

We will solve one more set of equations by this method, giving all the steps at length, the better to illustrate this notation.

$$\text{Given}\left\{\begin{array}{ll} 7x-2z+3u=17 & (1)\\ 4y-2z+\ t=11 & (2)\\ 5y-3x-2u=\ 8 & (3)\\ 4y-3u+2t\ =\ 9 & (4)\\ 3z+8u=33 & (5)\end{array}\right\}\text{, to find } x, y, z, u, t.$$

$$
\begin{array}{rl}
8y - 4z + 2t = 22 & (6) = (2) \times 2 \\
4y - 4z + 3u = 13 & (7) = (6) - (4) \\
21x - 6z + 9u = 51 & (8) = (1) \times 3 \\
35y - 21x - 14u = 56 & (9) = (3) \times 7 \\
35y - 6z - 5u = 107 & (10) = (8) + (9) \\
140y - 140z + 105u = 455 & (11) = (7) \times 35 \\
140y - 24z - 20u = 428 & (12) = (10) \times 4 \\
-116z + 125u = 27 & (13) = (11) - (12) \\
348z + 928u = 3828 & (14) = (5) \times 116 \\
-348z + 375u = 81 & (15) = (13) \times 3 \\
1303u = 3909 & (16) = (14) + (15) \\
u = 3 & (17) = (16) \div 1303 \\
8u = 24 & (18) = (17) \times 8 \\
3z = 9 & (19) = (5) - (18) \\
z = 3 & (20) = (19) \div 3 \\
3u = 9 & (21) = (17) \times 3 \\
4z = 12 & (22) = (20) \times 4 \\
4z - 3u = 3 & (23) = (22) - (21) \\
4y = 16 & (24) = (23) + (7) \\
y = 4 & (25) = (24) \div 4 \\
8y = 32 & (26) = (24) \times 2 \\
8y - 4z = 20 & (27) = (26) - (22) \\
2t = 2 & (28) = (6) - (27) \\
t = 1 & (29) = (28) \div 2 \\
2z = 6 & (30) = (20) \times 2 \\
3u - 2z = 3 & (31) = (21) - (30) \\
7x = 14 & (32) = (1) - (31) \\
x = 2 & (33) = (32) \div 7
\end{array}
$$

Collecting equations (33), (25), (20), (17), (29), we have

$$
\text{Ans.} \begin{cases} x = 2. \\ y = 4. \\ z = 3. \\ u = 3. \\ t = 1. \end{cases}
$$

ELIMINATION BY COMPARISON.

(81.) We may also eliminate one of the unknown quantities of two equations, by the following process:

Take the two equations

$$5y-4x=-22, \qquad (1)$$
$$4y+4x=38. \qquad (2)$$

If we, for a moment, consider y as a known quantity, we may then, from each of these equations, find the value of x by Rule under Art. 75.

We thus find

$$x=\frac{22+5y}{4}. \qquad (3)$$

$$x=\frac{38-3y}{4}. \qquad (4)$$

Putting these two values of x equal to each other, we have

$$\frac{22+5y}{4}=\frac{38-3y}{4}. \qquad (5)$$

Clearing (5) of fractions, it becomes

$$22+5y=38-3y, \qquad (6)$$

transposing and uniting terms, we find

$$8y=16 \therefore y=2.$$

This value of y substituted in either of the equations (3) or (4), will give

$$x=8.$$

The above method of eliminating may be given as in the following

RULE.

I. *Find, from each of the given equations, the value of one of the unknown quantities, by Rule under Art.* 75., *on the supposition that the other quantities are known.*

II. ***Then equate these different expressions of the value of the unknown, thus found, and we shall thus have a number of equations one less than were first given; and they will also contain a number of unknown quantities one less than at first.***

III. *Operating with these new equations as was done with the given equations, we can again reduce their number one; and continuing this process we shall finally have but one equation containing but one unknown quantity, which will then become known.*

EXAMPLES.

1. Given $\left\{\begin{array}{rl} 7x+5y+2z=\ \ 79 & (1) \\ 8x+7y+9z=122 & (2) \\ x+4y+5z=\ \ 55 & (3) \end{array}\right\}$, to find x, y, and z.

By Rule under Art. 75, we find, by using (1), (2) and (3),

$$x=\frac{79-5y-2z}{7}, \qquad (4)$$

$$x=\frac{122-7y-9z}{8}, \qquad (5)$$

$$x=55-4y-5z. \qquad (6)$$

Equating (4) and (6); and (5) and (6), we have

$$\frac{79-5y-2z}{7}=55-4y-5z, \qquad (7)$$

$$\frac{122-7y-9z}{8}=55-4y-5z. \qquad (8)$$

When cleared of fractions, (7) and (8) become

$$79-5y-2z=385-28y-35z,$$
$$122-7y-9z=440-32y-40z.$$

Transposing and uniting terms, we have

$$23y+33z=306, \qquad (9)$$
$$25y+31z=318. \qquad (10)$$

Equations (9) and (10) give

$$y=\frac{306-33z}{23}, \qquad (11)$$

$$y=\frac{318-31z}{25}. \qquad (12)$$

Equating (11) and (12), we have

$$\frac{306-33z}{23}=\frac{318-31z}{25}; \qquad (13)$$

which reduced gives

$$z=3.$$

This value of z substituted in (11) gives

$$y=9.$$

And these values of z and y, substituted in (6), give

$$x=4.$$

2. Given $\left\{\begin{array}{l} \frac{1}{2}x+\frac{1}{3}y+\frac{1}{4}z=62 \\ \frac{1}{3}x+\frac{1}{4}y+\frac{1}{5}z=47 \\ \frac{1}{4}x+\frac{1}{5}y+\frac{1}{6}z=38 \end{array}\right\}$, to find x, y, and z

These equations, when cleared of fractions, become

$$6x+\ 4y+\ 3z=\ 744, \qquad (1)$$
$$20x+15y+12z=2820, \qquad (2)$$
$$15x+12y+10z=2280. \qquad (3)$$

From (1), (2), and (3), we find

$$z=\frac{744-6x-4y}{3}, \qquad (4)$$

$$z=\frac{2820-20x-15y}{12}, \qquad (5)$$

$$z=\frac{2280-15x-12y}{10}. \qquad (6)$$

Equating (4) with (5), and (4), with (6), we have

$$\frac{744-6x-4y}{3}=\frac{2820-20x-15y}{12}, \qquad (7)$$

$$\frac{744-6x-4y}{3}=\frac{2280-15x-12y}{10}. \qquad (8)$$

Equations (7) and (8) when reduced become

$$4x+\ \ y=156, \qquad (9)$$

$$15x+4y=600. \qquad (10)$$

Equations (9) and (10) give

$$y=156-\ \ 4x, \qquad (11)$$

$$y=\frac{600-15x}{4}. \qquad (12)$$

Equating (11) and (12), we have

$$156-4x=\frac{600-15x}{4}. \qquad (13)$$

This reduced, gives

$$x=24.$$

Having found x, we readily find y and z to be

$$y=60;\ \ z=120.$$

ELIMINATION BY SUBSTITUTION.

(82.) There is still another method of elimination.

1. Suppose we have given the two equations

$$5x+2y=45, \qquad (1)$$

$$4x+\ \ y=33. \qquad (2)$$

From the first we find

$$y=\frac{45-5x}{2}. \qquad (3)$$

Substituting this value of y in (2), we have

$$4x+\frac{45-5x}{2}=33. \qquad (4)$$

Equation (4), when cleared of fractions, becomes

$$8x + 45 - 5x = 66. \qquad (5)$$

This gives

$$x = 7.$$

Substituting this value of x in (3), we find

$$y = 5.$$

2. Again, suppose we have given, to find x, y, and z, the three equations

$$2x + 4y - 3z = 22, \qquad (1)$$
$$4x - 2y + 5z = 18, \qquad (2)$$
$$6x + 7y - z = 63. \qquad (3)$$

From equation (3) we obtain

$$z = 6x + 7y - 63. \qquad (4)$$

Substituting this value of z, in (1) and (2), and they will become

$$2x + 4y - 3(6x + 7y - 63) = 22, \qquad (5)$$
$$4x - 2y + 5(6x + 7y - 63) = 18. \qquad (6)$$

Equations (5) and (6) become, after expanding, transposing, and uniting terms,

$$16x + 17y = 167, \qquad (7)$$
$$34x + 33y = 333. \qquad (8)$$

Equation (7) gives

$$x = \frac{167 - 17y}{16}. \qquad (9)$$

This value of x, substituted in (8), gives

$$\frac{34(167 - 17y)}{16} + 33y = 333. \qquad (10)$$

Equation (10), when solved as a simple equation of one un known quantity, gives

$$y = 7.$$

Substituting this value of y in (9), we find

$$z = 3.$$

Using these values of x and y in (4), we obtain

$$x = 4.$$

(83.) This method of eliminating may be comprehended in the following

RULE.

Having found the value of one of the unknown quantities, from either of the given equations, in terms of the other unknown quantities, substitute it for that unknown quantity in the remaining equations, and we shall thus obtain a new system of equations one less in number than those given. Operate with these new equations as with the first, and so continue until we find one single equation with but one unknown quantity, which will then become known.

EXAMPLES.

1. Given $\left.\begin{cases} x-w=50 & (1) \\ 3y-x=120 & (2) \\ 2z-y=120 & (3) \\ 3w-z=195 & (4) \end{cases}\right\}$, to find w, x, y, and z.

From (1) we find

$$w=x-50. \qquad (5)$$

This value of w, substituted in (4), gives

$$3(x-50)-z=195, \text{ or } 3x-z=345. \qquad (6)$$

Equation (6) gives

$$z=3x-345. \qquad (7)$$

This value of z, substituted in (3), gives

$$2(3x-345)-y=120, \text{ or } 6x-y=810 \qquad (8)$$

Equation (8) gives

$$y=6x-810. \qquad (9)$$

This value of y, substituted in (2), gives

$$3(6x-810)-x=120, \qquad (10)$$

or

$$17x=2550. \qquad (11)$$

$$\therefore x=150. \qquad (12)$$

This value of x causes (9) to become

$$y = 90.$$

Using the value of x in (7), we find

$$z = 105.$$

Finally, using the value of x in (5), we find

$$w = 100.$$

2. Given $\left\{\begin{array}{ll} x + \frac{1}{2}y = a, & (1) \\ y + \frac{1}{3}z = a, & (2) \\ z + \frac{1}{4}x = a, & (3) \end{array}\right\}$ to find x, y, and z.

Equation (3) gives

$$z = \frac{4a - x}{4}. \qquad (4)$$

This value of z, substituted in (2), we have

$$y + \frac{4a - x}{12} = a. \qquad (5)$$

Clearing of fractions and uniting terms, (5) becomes

$$12y - x = 8a. \qquad (6)$$

From (6) we find

$$x = 12y - 8a. \qquad (7)$$

This value of x, substituted in (1), gives

$$12y - 8a + \frac{y}{2} = a. \qquad (8)$$

Equation (8) gives

$$25y = 18a, \qquad (9)$$

Therefore,

$$y = \frac{18a}{25}.$$

This value of y, substituted in (7), gives

$$x = \frac{16a}{25}.$$

Substituting for x, in (4), its value just found, we have

$$z = \frac{21a}{25}.$$

Hence, collecting values, we have

$$\left.\begin{aligned} x &= \tfrac{16}{25}a. \\ y &= \tfrac{18}{25}a. \\ z &= \tfrac{21}{25}a. \end{aligned}\right\}$$

We may observe that if a is any multiple of 25, the above values of x, y, and z will be integers.

(84.) All equations of the first degree, containing any number of unknown quantities, can be solved by either of the Rules under Articles 79, 81, and 83, or by a combination of the same.

The student must exercise his own judgment, as to the choice of the above Rules. In very many cases he will discover many short processes, which depend upon the particular equations given.

(85.) We will now solve a few equations, and shall endeavor to effect their solution in the simplest manner possible.

1. Given $\left\{\begin{aligned} 6x + 5y &= 128, \\ 3x + 4y &= 88, \end{aligned}\right\}$ to find the values of x and y.

Adding the two equations, and dividing the sum by 9, we find

$$x + y = 24. \qquad (1)$$

Multiplying (1) by 3, and subtracting the result from the second of the given equations, we have

$$y = 16. \qquad (2)$$

Subtracting (2) from (1), we get

$$x = 8.$$

2. Given $\left\{\begin{aligned} x + y &= a, \quad (1) \\ y + z &= b, \quad (2) \\ z + x &= c, \quad (3) \end{aligned}\right\}$ to find x, y, and z.

Dividing the sum of these three equations by 2, we find

$$x+y+z=\frac{a+b+c}{2}. \qquad (4)$$

From (4) subtracting, successively, (2), (3), and (1), we find

$$\left.\begin{aligned} x&=\frac{b+c-b}{2},\\ y&=\frac{a-c+b}{2},\\ z&=\frac{-a+b+c}{2}. \end{aligned}\right\} \qquad (A)$$

Equations (1), (2), and (3), of this last example, are so related that if in (1) we change x to y, y to z, and a to b, it will correspond with (2). Again, if in (2) we change y to z, z to x and b to c, it will correspond with (3). Also, if in (3) we change z to x, x to y, and c to a, it will give (1), from which we first started. In each change we have advanced the letters one place lower in the alphabetical scale, observing that when we wish to change the last letters of the series, as z or c, we must change them respectively to x and a, the first of the series.

Since the above changes can be made with the primitive equations (1), (2), (3), without altering the conditions of the question, it follows that the same changes can be made in any of the equations derived from those. Thus, executing those changes in equations (A), we find that the first is changed into the second, the second into the third, and the third in turn is changed into the first.

3. Given $\left\{\begin{array}{ll}\dfrac{1}{x}+\dfrac{1}{y}=a, & (1)\\[2ex] \dfrac{1}{y}+\dfrac{1}{z}=b, & (2)\\[2ex] \dfrac{1}{z}+\dfrac{1}{x}=c, & (3)\end{array}\right\}$ to find x, y, and z

If we take the sum of these three equations, we shall obtain

$$\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=a+b+c. \qquad (4)$$

Now, subtracting twice (2) from (4), and we have

$$\frac{2}{x}=a+c-b. \qquad (5)$$

In a similar manner subtracting twice (3) and (1), successively, from (4), and we find

$$\frac{2}{y}=a-c+b; \qquad (6)$$

$$\frac{2}{z}=-a+b+c. \qquad (7)$$

Equations (5), (6), and (7), readily give

$$\left.\begin{array}{l} x=\dfrac{2}{a+c-b},\\[2ex] y=\dfrac{2}{a-c+b},\\[2ex] z=\dfrac{2}{-a+b+c}.\end{array}\right\} \qquad (B)$$

The letters in this example will admit of the same changes as those pointed out in the last example. Indeed, the only difference between the two examples is, that the unknown quantities in the one example are the reciprocals of those in the other. Consequently the expressions for x, y, and z,

as given by equations (B), ought to be the reciprocals of those given by equations (A), which we find to be really the case.

4. Given $\left\{\begin{array}{ll} x+a(y+z)=m, & (1) \\ y+b(x+z)=n, & (2) \\ z+c(x+y)=p, & (3) \end{array}\right\}$ to find x, y, and z.

If we add and subtract ax from the left-hand member of (1), and add and subtract by from the left-hand member of (2), and add and subtract cz from the left-hand member of (3), they will become

$$(1-a)x+a(x+y+z)=m, \qquad (4)$$
$$(1-b)y+b(x+y+z)=n, \qquad (5)$$
$$(1-c)z+c(x+y+z)=p. \qquad (6)$$

If we divide (4) by $1-a$, and (5) by $1-b$, and (6) by $1-c$, they will become

$$x+\frac{a}{1-a}(x+y+z)=\frac{m}{1-a}; \qquad (7)$$

$$y+\frac{b}{1-b}(x+y+z)=\frac{n}{1-b}; \qquad (8)$$

$$z+\frac{c}{1-c}(x+y+z)=\frac{p}{1-c}. \qquad (9)$$

Taking the sum of (7), (8), and (9), we have

$$\left\{\begin{array}{r} \left\{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}\right\}\times(x+y+z) \\ =\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}. \end{array}\right\} \qquad (10)$$

$$\text{Therefore, } x+y+z=\frac{\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}}{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}}. \qquad (11)$$

This value of $x+y+z$ substituted in (7), (8), and (9), gives

$$x=\frac{m}{1-a}-\frac{a}{1-a}\left\{\frac{\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}}{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}}\right\};\ (12)$$

$$y=\frac{n}{1-b}-\frac{b}{1-b}\left\{\frac{\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}}{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}}\right\};\ (13)$$

$$z=\frac{p}{1-c}-\frac{c}{1-c}\left\{\frac{\frac{m}{1-a}+\frac{n}{1-b}+\frac{p}{1-c}}{1+\frac{a}{1-a}+\frac{b}{1-b}+\frac{c}{1-c}}\right\}.\ (14)$$

This example affords a beautiful illustration of the law of permutations which can be made with the letters which enter into symmetrical equations. The primitive equations (1), (2), (3); the three equations (4), (5), and (6); and the three (7), (8), and (9); as well as the three (12), (13) and (14), can be deduced in succession from each other, by simply advancing the letters one place lower in the alphabetical scale. Equations (10) and (11), which contain all the different letters, are of such a form as not to change by this method of permuting. Consequently the expression within the braces of (12), (13), (14), which is the right-hand member of (11), must remain unchanged for the values of x, y, and z. By studying carefully the different laws by which changes may be made, we have great control over symmetrical algebraic expressions which we could not otherwise obtain. It is not always necessary that the change should be in alphabetical order, but may vary according to

any other law. The principle may be thus stated : whatever changes can be made among the letters entering into the primitive equations, without altering the equations, the same changes may be made on any of the derived equations.

This method of deducing one expression from another of a similar nature, is of great use, especially in the higher parts of analysis. In order that the proper permutations may be made with ease, and without danger of error, we must adopt some simple and uniform notation for the different values of the quantities which enter into our expressions. Indeed, by a well chosen method of notation, we may frequently resolve, with ease, questions which would otherwise be extremely difficult.

Perhaps we can not better impress upou the student, the importance of a judicious notation, than by giving, at length, the solution of the two following questions.

5. Find n numbers, such that the first increased by a_1 times the sum of all the others, shall equal b_1 ; the second, increased by a_2 times the sum of the others, equals b_2 ; the third, increased by a_3 times the sum of the rest, equals b_3 ; and so on for the other numbers.

SOLUTION.

Let the n numbers sought be represented by

$$x_1, x_2, x_3, \quad - \quad - \quad - \quad - \quad - \quad - \quad x_n.$$

Then, if

$$S = x_1 + x_2 + x_3 + \quad - \quad - \quad - \quad - \quad + x_n, \qquad (a)$$

we shall have, by the conditions, the following system of equations :

$$\left.\begin{array}{ll} x_1 + a_1 (S - x_1) = b_1, & (1) \\ x_2 + a_2 (S - x_2) = b_2, & (2) \\ x_3 + a_3 (S - x_3) = b_3, & (3) \\ - \quad - \quad - \quad - \quad - & - \\ - \quad - \quad - \quad - \quad - & - \\ x_n + a_n (S - x_n) = b_n. & (n) \end{array}\right\} \quad (A)$$

From (A) we readily find the following system of equations :

$$\left.\begin{array}{ll} x_1 = \dfrac{a_1}{a_1 - 1} \times S - \dfrac{b_1}{a_1 - 1}, & (1') \\ x_2 = \dfrac{a_2}{a_2 - 1} \times S - \dfrac{b_2}{a_2 - 1}, & (2') \\ x_3 = \dfrac{a_3}{a_3 - 1} \times S - \dfrac{b_3}{a_3 - 1}, & (3') \\ - \quad - \quad - \quad - \quad - & - \\ - \quad - \quad - \quad - \quad - & - \\ x_n = \dfrac{a_n}{a_n - 1} \times S - \dfrac{b_n}{a_n - 1}. & (n') \end{array}\right\} \quad (A')$$

Taking the sum of the n equations (A′), we find

$$S = D' \times S - D''. \qquad (B)$$

Where, for the sake of brevity, we have put

$$D' = \frac{a_1}{a_1 - 1} + \frac{a_2}{a_2 - 1} + \frac{a_3}{a_3 - 1} + - - - + \frac{a_n}{a_n - 1}, \quad (a')$$

$$D'' = \frac{b_1}{a_1 - 1} + \frac{b_2}{a_2 - 1} + \frac{b_3}{a_3 - 1} + - - - + \frac{b_n}{a_n - 1}. \quad (a'')$$

Returning to equation (B), we find

$$S = \frac{D''}{D' - 1}. \qquad (C)$$

This value of S written in the n equations (A′) gives

$$\left.\begin{aligned}
x_1 &= \frac{a_1}{a_1-1}\left\{\frac{D''}{D'-1}\right\}-\frac{b_1}{a_1-1},\\
x_2 &= \frac{a_2}{a_2-1}\left\{\frac{D''}{D'-1}\right\}-\frac{b_2}{a_2-1},\\
&- \quad - \quad - \quad - \quad - \quad - \quad -\\
&- \quad - \quad - \quad - \quad - \quad - \quad -\\
x_n &= \frac{a_n}{a_n-1}\left\{\frac{D''}{D'-1}\right\}-\frac{b_n}{a_n-1}.
\end{aligned}\right\} \quad (D)$$

If $n=10$; and $b_1=b_2=b_3=b_4=b_5=b_6=b_7=b_8=b_9=b_{10}=845693$; and $a_1=\frac{1}{3}$, $a_2=\frac{1}{5}$, $a_3=\frac{1}{7}$, $a_4=\frac{1}{9}$, $a_5=\frac{1}{9}$, $a_6=\frac{1}{10}$, $a_7=\frac{1}{11}$, $a_8=\frac{1}{12}$, $a_9=\frac{1}{13}$, $a_{10}=\frac{1}{14}$, then will the above question agree with one in the Higher Arithmetic, which question is there required to be solved by rules purely arithmetical. The preceding question is also a particular case of the above question.

6. Suppose n individuals, A_1, A_2, A_3, - - - - A_n, play together on this condition, that the one who loses shall give to each of the others as much as they then have. First A_1 loses, then A_2, then A_3, then A_4, and so on, until, in turn, they have all lost; and at the end of the nth game their respective shares are a_1, a_2, a_3, - - - - - a_n. How much had each before playing?

SOLUTION.

Let their respective shares before playing be represented by

$$x_1, x_2, x_3, \quad - \quad - \quad - \quad - \quad x_n.$$

Also, put

$$x_1+x_2+x_3+ \quad - \quad - \quad - \quad - \quad - \quad +x_n=S. \qquad (1)$$

Since A_1 loses on the first game, he must, by the question, give to A_2, A_3, A_4, &c., as much as they now have. Hence

A_1's money will be diminished by $x_2 + x_3 + x_4 + \text{- - - -} + x_n$, which, by (1), equals $S - x_1$, so that A_1's money will be

$$x_1 - (S - x_1) = 2x_1 - S.$$

Therefore, at the end of the first game, they will have

$$\begin{array}{llll} A_1, & A_2, & A_3, & A_n, \\ 2x_1 - S; & 2x_2; & 2x_3; \text{- - - - -} & 2x_n. \end{array}$$

Now, since A_2 loses on the second game, he must give to A_1, A_3, A_4, &c., as much as they now have. Hence A_2's money will be diminished by

$$2x_1 - S + 2x_3 + 2x_4 + \text{- - - - -} 2x_n. \qquad (2)$$

Since they, all together, always have the same amount as at first, we have

$$2x_1 - S + 2x_2 + 2x_3 + \text{- - - - -} 2x_n = S; \qquad (3)$$
$$\therefore 2x_1 - S + 2x_3 + 2x_4 + \text{- - - - -} 2x_n = S - 2x_2.$$

Hence, A_2, after the second game, will have

$$2x_2 - (S - 2x_2) = 4x_2 - S.$$

Therefore, at the end of the second game, they will have

$$\begin{array}{lllll} A_1, & A_2, & A_3, & A_4, & A_n, \\ 4x_1 - 2\,S; & 4x_2 - S; & 4x_3; & 4x_4; \text{- - - - -} & 4x_n. \end{array}$$

Proceeding in this way, we find that after the third game, they will have

$$\begin{array}{llllll} A_1, & A_2, & A_3, & A_4 & A_5, & A_n, \\ 8x_1 - 4\,S; & 8x_2 - 2\,S; & 8x_3 - S; & 8x_4; & 8x_5; \text{- - -} & 8x_n. \end{array}$$

And in general, after the nth game, they will have

$$\begin{array}{llll} A_1, & A_2, & A_3, & A_n, \\ 2^n x_1 - 2^{n-1} S; & 2^n x_2 - 2^{n-2} S; & 2^n x_3 - 2^{n-3} S; \text{- - -} & 2^n x_n - S. \end{array}$$

Equating these results with the values,

$$a_1,\ a_2,\ a_3,\ a_4,\ \text{- - - - -}\ a_n,$$

we have the n following equations:

$$\left.\begin{array}{ll}
2^n x_1 - 2^{n-1} S = a_1, & (1') \\
2^n x_2 - 2^{n-2} S = a_2, & (2') \\
2^n x_3 - 2^{n-3} S = a_3, & (3') \\
2^n x_4 - 2^{n-4} S = a_4, & (4') \\
- \quad - \quad - \quad - \quad - & - \\
- \quad - \quad - \quad - \quad - & - \\
2^n x_{n-1} - 2\,S = a_{n-1}, & ((n-1)') \\
2^n x_n - S = a_n, & (n')
\end{array}\right\} \quad (A)$$

From the above system of equations (A), we readily find

$$\left.\begin{array}{ll}
x_1 = \frac{a_1}{2^n} + \frac{S}{2}, & (1'') \\
x_2 = \frac{a_2}{2^n} + \frac{S}{2^2}, & (2'') \\
x_3 = \frac{a_3}{2^n} + \frac{S}{2^3}, & (3'') \\
- \quad - \quad - \quad - \quad - & - \\
- \quad - \quad - \quad - \quad - & - \\
x_{n-1} = \frac{a_{n-1}}{2^n} + \frac{S}{2^{n-1}}, & ([n-1]'') \\
x_n = \frac{a_n}{2^n} + \frac{S}{2^n}, & (n'')
\end{array}\right\} \quad (B)$$

Now, since they all together had as much money when they left off playing, as they had before playing, it follows, that

$$S = a_1 + a_2 + a_3 + a_4 + \; - \; - \; - \; - \; a_n. \quad (C)$$

If this value of S be substituted in the system of equations (B), we shall then have the values of

$$x_1, \; x_2, \; x_3, \; - \; - \; - \; - \; - \; x_n,$$

in terms of known quantities.

If we have the relation

$$a_1 = a_2 = a_3 = a_4 = \; - \; - \; - \; - \; = a_n,$$

then (C) will give

$$S = na_1,$$

and the system of equations (B) will then become

$$\left.\begin{aligned} x_1 &= a_1\left(\frac{n}{2} + \frac{1}{2^n}\right), \\ x_2 &= a_1\left(\frac{n}{2^2} + \frac{1}{2^n}\right), \\ x_3 &= a_1\left(\frac{n}{2^3} + \frac{1}{2^n}\right), \\ &- \; - \; - \; - \; - \\ &- \; - \; - \; - \; - \\ x_n &= a_1\left(\frac{n}{2^n} + \frac{1}{2^n}\right). \end{aligned}\right\} \qquad (D)$$

If, in (D), we suppose $n=5$ and $a_1=32$, we shall have

$$x_1 = 81\,;\; x_2 = 41\,;\; x_3 = 21\,;\; x_4 = 11\,;\; x_5 = 6.$$

The above supposition causes our question to agree with Ques. 13, Chap. XII, Higher Arithmetic.

7. Given $\left\{\begin{aligned} u+x+y&=13, \quad (1) \\ u+x+z&=17, \quad (2) \\ u+y+z&=18, \quad (4) \\ x+y+z&=21, \quad (4) \end{aligned}\right\}$ to find u, x, y, and z,

Dividing the sum of these four equations by 3, we obtain

$$u+x+y+z=23. \qquad (5)$$

From (5), subtracting successively, (4), (3), (2), and (1), and we find

$$\left.\begin{aligned} u &= 2, \\ x &= 5, \\ y &= 6, \\ z &= 10. \end{aligned}\right\}$$

8. Given $\left\{\begin{aligned} 3x+2y&=118, \\ x+5y&=191, \end{aligned}\right\}$ to find x and y.

Ans. $\left\{\begin{aligned} x&=16. \\ y&=35. \end{aligned}\right.$

9. Given $\left\{\begin{array}{c} 113\frac{1}{2}x - 27\frac{5}{7}y = 10y + 5488\frac{4}{7}, \\ 9y - 347 = 5x - 420, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x = 56. \\ y = 23 \end{array}\right.$

10. Given $\left\{\begin{array}{c} \dfrac{a}{b+y} = \dfrac{b}{3a+x}, \\ ax + 2by = d, \end{array}\right\}$ to find x and y.

Ans. $\left\{\begin{array}{l} x = \dfrac{2b^2 - 6a^2 + d}{3a}. \\ y = \dfrac{3a^2 - b^2 + d}{3b}. \end{array}\right.$

11. A and B possess together a fortune of \$570. If A's fortune were 3 times, and B's 5 times as great as each really is, then they would have together \$2350. How much had each?

Ans. A \$250, B \$320.

12. Find two numbers of the following properties: When the one is multiplied by 2, the other by 5, and both products added together, the sum is $= 31$; on the other hand, if the first be multiplied by 7, and the second by 4, and both products added together, we shall obtain 68.

Ans. The first is 8, and the second is 3.

13. A owes \$1200, B \$2550; but neither has enough to pay his debts. Lend me, said A to B, $\frac{1}{8}$ of your fortune, and I shall be enabled to pay my debts. B answered, I can discharge my debts, if you will lend me $\frac{1}{6}$ of yours. What was the fortune of each?

Ans. A's fortune is \$900, and that of B \$2400.

14. There is a fraction, such, that if 1 be added to the numerator, its value $= \frac{1}{3}$, and if 1 be added to the denominator, its value $= \frac{1}{4}$. What fraction is it?

Ans. $\frac{4}{15}$.

15. The sum of two numbers is $= a$, the quotient arising from the division of the second, by the first is $= b$. Find these numbers?

Ans. $\frac{a}{b+1}$, and $\frac{ab}{b+1}$.

16. A, B, C, owe together \$2190, and none of them can alone pay this sum; but when they unite, it can be done in the following ways: first, by B's putting $\frac{3}{7}$ of his property to all of A's; secondly, by C's putting $\frac{5}{9}$ of his property to all B's; or, by A's adding $\frac{2}{3}$ of his property to that of C. How much did each possess?

Ans. A \$1530; B \$1540; and C \$1170.

17. A and B possess, together, only $\frac{2}{3}$ of the property of C; B and C have, together, 6 times as much as A; were B \$680 richer than he actually is, then he would have as much as A and C together. How much has each?

Ans. A, has \$200; B, \$360; and C, \$840.

18. Three masons, A, B, C, are to build a wall. A and B, jointly, could build this wall in 12 days; B and C could accomplish it in 20 days; but C and A would do it in 15 days. What time would each take to do it alone in? And in what time will they finish it, if all three work together?

Ans. A requires 20 days, B 30, and C 60;
all three together require 10 days.

19. Three laborers are employed in a certain work. A and B would, together, complete this work in a days; B and C require b days; but C and A, only c days. What time would each require, singly, to accomplish it in? And in what time would they finish it, if they all three worked together?

Answer,

A requires $\frac{2abc}{ab+bc-ca}$ days, B, $\frac{2abc}{bc+ca-ab}$ days,

C, $\frac{2abc}{ca+ab-bc}$ days; Jointly, $\frac{2abc}{ab+bc+ca}$ days.

20. A certain number consists of three digits, which are in an arithmetical progression. If this number be divided by the sum of its digits, (that is, without considering the value they have as tens and hundreds,) the quotient is 48; but if 198 be subtracted from it, then we obtain for the remainder a number consisting of the same digits as the one sought, but in an inverted order. What number is this?

Ans. 432.

21. A cistern containing 210 buckets, may be filled by 2 pipes. By an experiment, in which the first was open 4, and the second 5 hours, 90 buckets of water were obtained. By another experiment, when the first was open 7, and the other $3\frac{1}{2}$ hours, 126 buckets were obtained. How many buckets does each pipe discharge in an hour. And in what time will the cistern be filled, when the water flows from both pipes at once?

Ans. The first pipe discharges 15, and the second, 6 buckets; it will require 10 hours for them to fill the cistern.

22. According to Vitruvius, Hiero's crown weighed 20 lbs., and lost $1\frac{1}{4}$ lbs., nearly, in water. Let it be assumed that it consisted of gold and silver only, and that 20 lbs. of gold lose 1 lb. in water, and 10 lbs. of silver, in like manner, lose 1 lb. How much gold, and how much silver did this crown contain.

Ans. 15 lbs. of gold, and 5 pounds of silver.

23. A person has two large pieces of iron whose weight is required. It is known that $\frac{2}{5}$ of the first piece weighs 96 lbs. less than $\frac{3}{4}$ of the other piece; and that $\frac{5}{8}$ of the other piece weighs exactly as much as $\frac{4}{9}$ of the first. How much did each of these pieces weigh?

Ans. The first weighs 720 lbs., the second 512 lbs.

24. Two persons, A and B, can together perform a piece

of work in 16 days. After having laboured jointly 4 days, A leaves, and B by laboring 36 days more, completes it. How many days would each separately require ?

$$\text{Ans.} \begin{cases} \text{A requires 24 days,} \\ \text{B requires 48 days.} \end{cases}$$

25. A merchant has two kinds of wine ; if he mix a gallons of the worst wine with b of the best, the mixture is worth c dollars per gallon ; but if he mix f gallons of the worst with g gallons of the best, then the mixture is worth h dollars per gallon. What is the price of each kind of wine per gallon ?

$$\text{Ans.} \begin{cases} \text{Price of the worst, } \dfrac{(a+b)\,cg-(f+g)\,bh}{ag-bf}, \\ \text{Price of the best, } \dfrac{(a+b)\,cf-(f+g)\,ah}{bf-ag}. \end{cases}$$

26. Several detachments of artillery divided a certain number of cannon balls. The first took 72 and $\frac{1}{9}$ of the remainder ; the next 144 and $\frac{1}{9}$ of the remainder ; the third 216 and $\frac{1}{9}$ of the remainder ; and the fourth 288 and $\frac{1}{9}$ of the remainder, and so on ; when it was found that the balls had been equally divided. What was the number of detachments and the number of balls ?

Ans. 8 detachments, and 4608 balls

27. A person has three horses and a saddle, which of itself is worth 220 dollars. If he put the saddle on the back of the first horse, it will make his value equal to that of the second and third ; but if he put it on the back of the second horse, it will make his value double that of the first and third ; and if he put it on the back of the third horse, it will make his value triple that of the first and second. What is the value of each horse ?

Ans. 20, 100, and 140 dollars.

ELIMINATION BY INDETERMINATE MULTIPLIERS.

(86.) Suppose we wish to find x and y from the equations

$$2x + 3y = 13, \tag{1}$$
$$5x + 4y = 22. \tag{2}$$

Multiplying (1) by m, we find

$$2mx + 3my = 13m. \tag{3}$$

Adding (2) and (3), we have

$$(2m + 5)x + (3m + 4)y = 13m + 22. \tag{4}$$

Assume $3m + 4 = 0$ which gives

$$m = -\tfrac{4}{3}. \tag{5}$$

This value of m causes equation (4) to become

$$x = \frac{13m + 22}{2m + 5} = 2. \tag{6}$$

Again, if we had assumed $2m + 5 = 0$, which would have given

$$m = -\tfrac{5}{2}, \tag{7}$$

then equation (4) would have become

$$y = \frac{13m + 22}{3m + 4} = 3. \tag{8}$$

Now, returning to our former equations, we will subtract (2) from (3); we thus obtain

$$(2m - 5)x + (3m - 4)y = 13m - 22. \tag{9}$$

Assume $3m - 4 = 0$, which gives

$$m = \tfrac{4}{3}. \tag{10}$$

This value of m causes (9) to become

$$x = \frac{13m - 22}{2m - 5} = 2. \tag{11}$$

Again, assume $2m - 5 = 0$, which gives

$$m = \tfrac{5}{2}. \tag{13}$$

This causes (9) to become

$$y = \frac{13m + 22}{3m - 4} = 3. \qquad (13)$$

These values of x and y are the same as just found.

It is evident that had we multiplied (2) by m, and then added, or subtracted the result from (1), we should then have found, in a similar manner, the same values for x and y.

(87.) We will now apply this method to the two literal equations,

$$X_1x + Y_1y = A_1, \qquad (1)$$
$$X_2x + Y_2y = A_2. \qquad (2)$$

In these equations the capital letters are supposed to be known, and their *subscript* numerals indicate the equation to which they belong. Thus,

X_2 is the coefficient of x in the second equation.

Y_1 is the coefficient of y in the first equation.

A_2 is the absolute term, or the term independent of x and y in the second equation.

Returning to our equations, we will multiply (1) by m and add the result to (2); we thus obtain

$$(X_1m + X_2)x + (Y_1m + Y_2)y = A_1m + A_2. \qquad (3)$$

Assume $Y_1m + Y_2 = 0$, which gives

$$m = -\frac{Y_2}{Y_1}. \qquad (4)$$

This causes (3) to become

$$(X_1m + X_2)x = A_1m + A_2, \qquad (5)$$

which gives immediately

$$x = \frac{A_1m + A_2}{X_1m + X_2} = \frac{A_1Y_2 - A_2Y_1}{X_1Y_2 - X_2Y_1}. \qquad (6)$$

Assume $X_1m + X_2 = 0$, which gives

$$m = -\frac{X_2}{X_1}. \qquad (7)$$

This value of m causes (3) to become

$$y = \frac{A_1 m + A_2}{Y_1 m + Y_2} = \frac{A_2 X_1 - A_1 X_2}{X_1 Y_2 - X_2 Y_1}. \qquad (8)$$

Hence, the values of x and y are

$$\left.\begin{aligned} x &= \frac{A_1 Y_2 - A_2 Y_1}{X_1 Y_2 - X_2 Y_1}, \\ y &= \frac{A_2 X_1 - A_1 X_2}{X_1 Y_2 - X_2 Y_1}. \end{aligned}\right\} \qquad (9)$$

These values of x and y may be considered as comprising the solution of all simple equations combining only two unknown quantities. If we wish to adapt this general solution to the equations

$$2x + 3y = 13,$$
$$5x + 4y = 22;$$

we must call

$$A_1 = 13; \; A_2 = 22.$$
$$X_1 = 2; \; X_2 = 5.$$
$$Y_1 = 3; \; Y_2 = 4.$$

These values substituted in (9), give

$$x = 2; \; y = 3.$$

(88.) As a still farther illustration of the method of elimination by means of indeterminate multipliers, we will proceed to the solution of three simultaneous simple equations, involving three unknown quantities x, y and z; and we will continue to make use of the notation by the assistance of subscript numbers.

Let the equations be as follows:

$$X_1x + Y_1y + Z_1z = A_1, \qquad (1)$$
$$X_2x + Y_2y + Z_2z = A_2, \qquad (2)$$
$$X_3x + Y_3y + Z_3z = A_3. \qquad (3)$$

In these equations, as in those of the last example, the capital letters X, Y, Z, are the coefficients of their corresponding small letters. The small numerals placed at the base of these coefficients correspond to the particular equation to which they belong. Thus X_2 is the coefficient of x in the second equation; Y_3 is the coefficient of y in the third equation; Z_1 is the coefficient of z in in the first equation, and so for the other coefficients. The letter A is used to denote the right-hand members of the equations, or the absolute terms; the subscript numbers in this case also denote the equation to which they belong.

This kind of notation, by use of subscript numbers, is very natural and simple, and combines many advantages over the ordinary methods.

Having explained this method of notation, we will now proceed to the solution of our equations.

If we multiply (1) by m, and (2) by n, and then add the results, we shall obtain

$$\left.\begin{array}{l}(X_1m + X_2n)x + (Y_1m + Y_2n)y \\ +(Z_1m + Z_2n)z = A_1m + A_2n.\end{array}\right\} \qquad (4)$$

From (4) subtracting (3), we find

$$\left.\begin{array}{l}(X_1m+X_2n-X_3)x+(Y_1m+Y_2n-Y_3)y \\ +(Z_1m+Z_2n-Z_3)z=A_1m+A_2n-A_3.\end{array}\right\} \qquad (5)$$

In order to cause y and z to vanish from this equation, we will assume

$$Y_1m + Y_2n = Y_3, \qquad (6)$$
$$Z_1m + Z_2n = Z_3. \qquad (7)$$

This assumption causes (5) to become

$$(X_1m + X_2n - X_3)x = A_1m + A_2n - A_3. \tag{8}$$

Therefore,

$$x = \frac{A_1m + A_2n - A_3}{X_1m + X_2n - X_3}. \tag{9}$$

We must now find the values of m and n, by aid of conditions (6) and (7); for this purpose we will compare them with (1) and (2), (Art. 87). Now, in order to make (6) and (7) agree with (1) and (2) respectively, we must change x to m, y to n; X_1 to Y_1, X_2 to Z_1, Y_1 to Y_2, Y_2 to Z_2, A_1 to Y_3, A_2 to Z_3. Making these same changes in equations (9) of Art. 87, we obtain

$$m = \frac{Y_3Z_2 - Z_3Y_2}{Y_1Z_2 - Z_1Y_2}. \tag{10}$$

$$n = \frac{Z_3Y_1 - Y_3Z_1}{Y_1Z_2 - Z_1Y_2}. \tag{11}$$

Arranging the terms alphabetically, we have

$$m = \frac{Y_3Z_2 - Y_2Z_3}{Y_1Z_2 - Y_2Z_1}. \tag{12}$$

$$n = \frac{Y_1Z_3 - Y_3Z_1}{Y_1Z_2 - Y_2Z_1}. \tag{13}$$

Substituting these values of m and n, in (9), we readily find

$$x=\frac{\mathcal{A}_1\,Y_3\,Z_2-\mathcal{A}_1\,Y_2\,Z_3+\mathcal{A}_2\,Y_1\,Z_3-\mathcal{A}_2\,Y_3\,Z_1+\mathcal{A}_3\,Y_2\,Z_1-\mathcal{A}_3\,Y_1\,Z_2}{X_1\,Y_3\,Z_2-X_1\,Y_2\,Z_3+X_2\,Y_1\,Z_3-X_2\,Y_3\,Z_1+X_3\,Y_2\,Z_1-X_3\,Y_1\,Z_2}. \qquad (14)$$

If we change the signs of all the terms of the numerator and denominator, and make a slight change in their arrangement, we shall have

$$x=\frac{\mathcal{A}_1\,Y_2\,Z_3+\mathcal{A}_2\,Y_3\,Z_1+\mathcal{A}_3\,Y_1\,Z_2-\mathcal{A}_1\,Y_3\,Z_2-\mathcal{A}_2\,Y_1\,Z_3-\mathcal{A}_3\,Y_2\,Z_1}{X_1\,Y_2\,Z_3+X_2\,Y_3\,Z_1+X_3\,Y_1\,Z_2-X_1\,Y_3\,Z_2-X_2\,Y_1\,Z_3-X_3\,Y_2\,Z_1}. \qquad (15)$$

By a similar process we shall find the values of y and z, as below.

$$y=\frac{X_1\,\mathcal{A}_2\,Z_3+X_2\,\mathcal{A}_3\,Z_1+X_3\,\mathcal{A}_1\,Z_2-X_1\,\mathcal{A}_3\,Z_2-X_2\,\mathcal{A}_1\,Z_3-X_3\,\mathcal{A}_2\,Z_1}{X_1\,Y_2\,Z_3+X_2\,Y_3\,Z_1+X_3\,Y_1\,Z_2-X_1\,Y_3\,Z_2-X_2\,Y_1\,Z_3-X_3\,Y_2\,Z_1}. \qquad (16)$$

$$z=\frac{X_1\,Y_2\,\mathcal{A}_3+X_2\,Y_3\,\mathcal{A}_1+X_3\,Y_1\,\mathcal{A}_2-X_1\,Y_3\,\mathcal{A}_2-X_2\,Y_1\,\mathcal{A}_3-X_3\,Y_2\,\mathcal{A}_1}{X_1\,Y_2\,Z_3+X_2\,Y_3\,Z_1+X_3\,Y_1\,Z_2-X_1\,Y_3\,Z_2-X_2\,Y_1\,Z_3-X_3\,Y_2\,Z_1}. \qquad (17)$$

(89). We will now proceed to point out some remarkable relations in the combinations of the letters marked with subscript numbers, as given by equations (15), (16), and (17).

I. The denominator, which is common to the three expressions, is composed of six distinct products, each consisting of three independent factors. Three of these products are positive, and three are negative.

II. The letters forming the different products of this common denominator being always arranged in alphabetical order, X, Y, Z, we remark that the subscript numbers of the first product are 1, 2, 3. Now, if we add a unit to each of these numbers, observing that when the sum becomes 4 to substitute 1, we shall obtain 2, 3, 1, which are the subscript numbers of the second product. Again, increasing each of these by 1, observing as before, to write 1 when the sum becomes 4, we find 3, 1, 2, which are the subscript numbers of the third product. If we increase each of these last numbers by 1, observing the same law, we shall obtain 1, 2, 3, which are the subscript numbers belonging to the first product. A similar method of changing has already been noticed under Art. 85.

What we have said in regard to the subscript numbers of the positive products, applies equally well in respect to the negative products.

III. The numerator of the expression for x, may be derived from the common denominator by simply substituting A for X, observing to retain the same subscript numbers.

The numerator of the expression for y may be derived from the common denominator by substituting A for Y, observing to retain the same subscript numbers.

In the same way may the numerator of the expression for z be found by changing Z of the denominator into A, retaining the same subscript numbers.

(90.) We will now proceed to show how these expressions, for x, y, and z, can be obtained by a very simple and novel process, which is easily retained in the memory, and which is applicable to all simple equations involving only three unknown quantities.

Writing the coefficients and the absolute terms in the same order as they are now placed in equations (1), (2), (3), we have

$$\left.\begin{array}{llll} X_1 & Y_1 & Z_1 = & \mathcal{A}_1, \\ X_2 & Y_2 & Z_2 = & \mathcal{A}_2, \\ X_3 & Y_3 & Z_3 = & \mathcal{A}_3. \end{array}\right\} \qquad (\mathcal{A})$$

Now, all the products of the common denominator can be found by multiplying together by threes, the coefficients which are found by passing obliquely from the left to the right, observing that if the products obtained by passing obliquely *downwards*, are taken positively, then those formed by passing obliquely *upwards* must be taken negatively, and conversely. This is in accordance with the property of the negative sign: In the present case the products formed by passing obliquely *downwards*, are taken positive.

In this sort of checker-board movement, we must observe that when we run out at the bottom of any column, we must pass to the top of the same column; and when we run out at the top, we must pass to the bottom of the same column.

This method is most readily performed upon the black-board, by drawing oblique lines connecting the successive factors of the different products.

We will trace out this sort of oblique movement.

Commencing with X_1, we pass obliquely downwards to Y_2, and thence to Z_3, and thus obtain the positive product of $X_1\,Y_2\,Z_3$.

Commencing with X_2, we pass obliquely downwards to Y_3, and since we have now run out with the column of Z's at the bottom, we pass to Z_1, at the top of the column, and thus obtain the positive product $X_2Y_3Z_1$.

Again, commencing with X_3, we pass to Y_1, and thence obliquely downwards to Z_2, and find the positive product $X_3Y_1Z_2$.

Now, for the negative products we make similar movements obliquely upwards.

Thus, commencing with X_1, we pass to Y_3, and thence obliquely upwards to Z_2, and find the negative product $X_1Y_3Z_2$.

Commencing with X_2, we pass obliquely upwards to Y_1, and thence to Z_3, and find the negative product $X_2\,Y_1\,Z_3$.

Again, commencing with X_3, we pass obliquely upwards to Y_2, and thence to Z_1, and thus obtain the negative product $X_3Y_2Z_1$.

Having thus obtained the denominator which is common to the values of x, y, z; we may find the numerator of the value of x, by supposing the $\mathcal{A}$'s to take the place of the X's, and then to repeat our checker-board movement. By changing the Y's into the $\mathcal{A}$'s, we shall find the numerator of the value of y; and by changing the Z's into $\mathcal{A}$'s we shall find the numerator of the value of z.

(91.) We will now illustrate this method of solving simple equations containing only three unknown quantities, by a few examples.

1. Given $\left\{\begin{array}{l} 2x + 3y + 4z = 16, \\ 3x + 5y + 7z = 26, \\ 4x + 2y + 3z = 19, \end{array}\right\}$ to find x, y, and z.

We will first find the common denominator.

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times5\times3 = 30$	$2\times2\times7 = -28$
$3\times2\times4 = 24$	$3\times3\times3 = -27$
$4\times3\times7 = 84$	$4\times5\times4 = -80$
138	-135
-135	

$3 =$ common denominator.

We have for the numerator of x the following operation:

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$16\times5\times3 = 240$	$16\times2\times7 = -224$
$26\times2\times4 = 208$	$26\times3\times3 = -234$
$19\times3\times7 = 399$	$19\times5\times4 = -380$
847	-838
-838	

$9 =$ numerator, for x.

To find the numerator for y, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$2\times26\times3 = 156$	$2\times19\times7 = -266$
$3\times19\times4 = 228$	$3\times16\times3 = -144$
$4\times16\times7 = 448$	$4\times26\times4 = -416$
832	-826
-826	

$6 =$ numerator, for y.

To find the numerator for z, we have

Positive Products.	Negative Products.
$2\times5\times19=190$	$2\times2\times26=-104$
$3\times2\times16=\ \ 96$	$3\times3\times19=-171$
$4\times3\times26=312$	$4\times5\times16=-320$
598	-595
-595	
$3=$ numerator, for z.	

Hence,

$$x=\tfrac{9}{3}=3.$$

$$y=\tfrac{6}{3}=2.$$

$$z=\tfrac{3}{3}=1.$$

When some of the coefficients are negative, we must observe the rule for the multiplication of signs.

2. Given $\left\{\begin{array}{l} 2x+4y-3z=22, \\ 4x-2y+5z=18, \\ 6x+7y-\ \ z=62, \end{array}\right\}$ to find x, y, and z.

To find the common denominator, we have

Positive Products.	Negative Products.
$2\times-2\times-1=\ \ 4$	$2\times\ 7\times\ 5=-70$
$4\times\ 7\times-3=-84$	$4\times\ 4\times-1=\ \ 16$
$6\times\ 4\times\ 5=\ 120$	$6\times-2\times-3=-36$
40	-90
-90	
$-50=$ common denominator.	

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$22\times-2\times-1=\quad 44$	$22\times\ 7\times\ 5=-770$
$18\times\ 7\times-3=-378$	$18\times\ 4\times-1=\quad 72$
$63\times\ 4\times\ 5=\ 1260$	$63\times-2\times-3=-378$
926	-1076
-1076	
$-150=$ numerator, for x.	

Hence, $x=\dfrac{-150}{-50}=3.$

Proceeding in a similar way, we find the values of y and z.

3. Given $\left\{\begin{array}{l} x+\frac{1}{2}y=a, \\ y+\frac{1}{3}z=a, \\ z+\frac{1}{4}x=a, \end{array}\right\}$ to find x, y, and z.

We will arrange the coefficients, omitting the unknown quantities, observing also to write 0 for such terms as are wanting.

This arrangement being made, we have

$$\begin{array}{cccll} 1 & \frac{1}{2} & 0 & = & a \\ 0 & 1 & \frac{1}{3} & = & a \\ \frac{1}{4} & 0 & 1 & = & a \end{array}$$

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$1\times1\times1=\ 1$	$1\times0\times\frac{1}{3}=0$
$0\times0\times0=\ 0$	$0\times\frac{1}{2}\times1=0$
$\frac{1}{4}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{24}$	$\frac{1}{4}\times1\times0=0$
$\frac{25}{24}=$ common denominator.	0

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$a\times1\times1=a$	$a\times0\times\frac{1}{3}=0$
$a\times0\times0=0$	$a\times\frac{1}{2}\times1=-\frac{1}{2}a$
$a\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{6}a$	$a\times1\times0=0$
$\frac{7}{6}a$	$-\frac{1}{2}a$
$-\frac{1}{2}a$	

$$\frac{2}{3}a=\text{numerator, for } x.$$

Hence, $x=\frac{2}{3}a\div\frac{25}{24}=\frac{16}{25}a.$

By a similar process is the value of y and z found.

4. Given $\left\{\begin{array}{l} x+a(y+z)=m, \\ y+b(x+z)=n, \\ z+c(x+y)=p, \end{array}\right\}$ to find x, y, and z.

These coefficients, being properly arranged give,

$$\begin{array}{cccccc} 1 & a & a & = & m \\ b & 1 & b & = & n \\ c & c & 1 & = & p \end{array}$$

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$1\times1\times1=1$	$1\times c\times b=-bc$
$b\times c\times a=abc$	$b\times a\times1=-ab$
$c\times a\times b=abc$	$c\times1\times a=-ac$
$1+2abc$	$-ab-ac-bc$
$-ab-ac-bc$	

$1+2abc-ab-ac-bc=$ common denominator.

For the numerator of x, we have

POSITIVE PRODUCTS.	NEGATIVE PRODUCTS.
$m\times 1\times 1 = m$	$m\times c\times b = -bcm$
$n\times c\times a = acn$	$n\times a\times 1 = -an$
$p\times a\times b = abp$	$p\times 1\times a = -ap$
$m + acn + abp$	$-bcm - an - ap$
$-bcm - an - ap$	

$$m+acn+abp-bcm-an-ap = \text{numerator of } x.$$

Hence, $$x=\frac{m+acn+abp-bcm-an-ap}{1+2abc-ab-ac-bc}.$$

If to this expression for x we apply the principle of permutation, as already explained, by advancing the letters one place lower in the alphabetical scale, we shall find

$$y=\frac{n+bap+bcm-can-bp-bm}{1+2bca-bc-ba-ca}.$$

Again, permuting this expression, we have

$$z=\frac{p+cbm+can-abp-cm-cn}{1+2cab-ca-cb-ab}.$$

This solution is far shorter than the one given on page 94, and the expressions for x, y, and z, are far more simple.

We may remark, that the denominators of the above expressions are common, as they must of necessity be, in virtue of the general results given by Equations (15), (16), (17), on page 111.

5. A, B, and C, owe together (a) \$2190, and none of them can alone pay this sum; but when they unite, it can be done in the following ways: first, by B's putting $\frac{3}{7}$ of his property to all of A's; secondly, by C's putting $\frac{5}{9}$ of his property to all of B's; or by A's putting $\frac{2}{3}$ of his property to all of C's. How much was each worth?

Let x, y, and z, represent what A, B, and C, were respectively worth.

Then we shall have these conditions,

$$x + \tfrac{3}{7}y = a,$$
$$y + \tfrac{5}{9}z = a,$$
$$z + \tfrac{2}{3}x = a.$$

Clearing these of fractions, and arranging the coefficients, we have

$$\begin{array}{ccccc} 7 & 3 & 0 & = & 7a \\ 0 & 9 & 5 & = & 9a \\ 2 & 0 & 3 & = & 3a \end{array}$$

Positive Products.	Negative Products.
$7 \times 9 \times 3 = 189$	$7 \times 0 \times 5 = 0$
$0 \times 0 \times 0 = 0$	$0 \times 3 \times 3 = 0$
$2 \times 3 \times 5 = 30$	$2 \times 9 \times 0 = 0$
219 = common denominator.	0

For the numerator of x, we have

Positive Products.	Negative Products.
$7a \times 9 \times 3 = 189a$	$7a \times 0 \times 5 = 0$
$9a \times 0 \times 0 = 0$	$9a \times 3 \times 3 = -81a$
$3a \times 3 \times 5 = 45a$	$3a \times 9 \times 9 = 0$
$234a$	$-81a$
$-81a$	
$153a$ = numerator of x.	

Hence, $x = \dfrac{153a}{219} = \dfrac{153 \times 2190}{219} = 1530.$

For the numerator of y, we find

POSITIVE PRODUCTS.

$$\begin{aligned}7\times 9a\times 3 &= 189a\\ 0\times 3a\times 0 &= 0\\ 2\times 7a\times 5 &= 70a\\ \hline & \quad 259a\\ & -105a\\ \hline & \quad 154a = \text{numerator of } y.\end{aligned}$$

NEGATIVE PRODUCTS

$$\begin{aligned}7\times 3a\times 5 &= -105a\\ 0\times 7a\times 3 &= 0\\ 2\times 9a\times 0 &= 0\\ \hline & -105a\end{aligned}$$

Hence, $y = \dfrac{154a}{219} = 1540.$

For the numerator of z, we have

POSITIVE PRODUCTS.

$$\begin{aligned}7\times 9\times 3a &= 189a\\ 0\times 0\times 7a &= 0\\ 2\times 3\times 9a &= 54a\\ \hline & \quad 243a\\ & -126a\\ \hline & \quad 117a = \text{numerator of } z.\end{aligned}$$

NEGATIVE PRODUCTS.

$$\begin{aligned}7\times 0\times 9a &= 0\\ 0\times 3\times 3a &= 0\\ 2\times 9\times 7a &= -126a\\ \hline & -126a\end{aligned}$$

Hence, $z = \dfrac{117a}{219} = 1170.$

Collecting the results, we find that

A was worth \$1530,
B " " \$1540,
C " " \$1170.

The student will find, after a little practice in this method that it is much more simple than would at first sight seem.

Whenever some of the coefficients are zeros, as in the 3d and 5th examples, the work is much abridged, as in this caee some of the products must become zero.

CHAPTER IV.

INVOLUTION, EVOLUTION, IRRATIONAL AND IMAGINARY QUANTITIES.

INVOLUTION.

(92.) The process of raising a quantity to any proposed power is called INVOLUTION.

When the quantity to be involved is a single letter, it is involved by placing the number denoting the power above it a little to the right. (Art. 11.)

After the same manner we may represent the power of any quantity, by enclosing it within a parenthesis, and then treating it as a single letter.

Thus,

the second power of $mx = (mx)^2$,
the third power of $a + b = (a + b)^3$,
the fourth power of $3m + y = (3m + y)^4$,
&c., &c.

CASE I.

(93.) To involve a monomial, we obviously have this

RULE.

I. *Raise the coefficient to the required power, by actual multiplication.*

II. *Raise the different letters to the required power by multiplying the exponents, which they already have, by the number denoting the power, observing that if no exponent is written, then one is always understood. To this power prefix the power of the coefficient.*

NOTE.—If the quantity to be involved is negative, the signs of the *even* powers must be positive, and the signs of the *odd* powers negative. (Art. 29.)

EXAMPLES.

1. What is the square of $3ax^3$?

Here the square of 3 equals

$$3^2 = 3 \times 3 = 9.$$

Considering the exponent of a, in the expression ax^3, as one, we find a^2x^6 for the square of ax^3.

Therefore we have

$$(3ax^3)^2 = 9a^2x^6.$$

2. What is the fifth power of $-2ab^3$?

Ans. $(-2ab^3)^5 = -32a^5b^{15}$.

3. What is the fourth power of $-\frac{1}{3}xy^{-2}$?

Ans. $(-\frac{1}{3}xy^{-2})^4 = \frac{1}{81}x^4y^{-8}$,

which by Art. 49, is the same as

$$\frac{x^4}{81y^8}.$$

4. What is the seventh power of $-a^{-\frac{1}{3}}x$?

Ans. $-a^{-\frac{7}{3}}x^7=-\dfrac{x^7}{a^{\frac{7}{3}}}$.

5. What is the third power of x^3y^{-1}?

Ans. $x^9y^{-3}=\dfrac{x^9}{y^3}$.

6. What is the nth power of $-2x^{-3}y^2$?

Ans. $\pm 2^nx^{-3n}y^{2n}=\pm\dfrac{2^ny^{2n}}{x^{3n}}$.

7. What is the square of $-7x^{-1}y^{-3}$?

Ans. $49x^{-2}y^{-6}=\dfrac{49}{x^2y^6}$.

8. What is the third power of $-\dfrac{1}{5}x^3y^{-5}$?

Ans. $-\dfrac{1}{125}x^9y^{-15}=-\dfrac{x^9}{125y^{15}}$.

9. What is the seventh power of $-m^{\frac{1}{3}}xz^{-1}$?

Ans. $-m^{\frac{7}{3}}x^7z^{-7}$.

10. What is the fourth power of $-\dfrac{2}{3}n^{-2}y^3$?

Ans. $\dfrac{16}{81}n^{-8}y^{12}$.

CASE II.

(94) When the quantity is compound, we can write the different powers by the aid of rules which we will hereafter point out. (See Binomial Theorem.)

At present we will content ourselves, by involving compound expressions by actual multiplication, according to Rule under Art. 33.

EXAMPLES.

1. Find the second power of $x+y-z$.

$$\begin{array}{l}
x+y-z \\
x+y-z \\
\hline
x^2+xy-xz \\
\quad +xy \qquad +y^2-yz \\
\qquad\quad -xz \qquad -yz+z^2 \\
\hline
\end{array}$$

Ans. $=x^2+2xy-2xz+y^2-2yz+z^2$.

2. Find the fifth power of $a+b$, as well as all the lower powers of the same.

$$(a+b)^1=a+b.$$

$$\begin{array}{l}
a+b \\
\hline
a^2+ab \\
\quad +ab+b^2 \\
\hline
\end{array}$$

$$(a+b)^2=a^2+2ab+b^2.$$

$$\begin{array}{l}
a+b \\
\hline
a^3+2a^2b+ab^2 \\
\qquad a^2b+2ab^2+b^3 \\
\hline
\end{array}$$

$$(a+b)^3=a^3+3a^2b+3ab^2+b^3.$$

$$\begin{array}{l}
a+b \\
\hline
a^4+3a^3b+3a^2b^2+ab^3 \\
\qquad a^3b+3a^2b^2+3ab^3+b^4 \\
\hline
\end{array}$$

$$(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4.$$

$$\begin{array}{l}
a+b \\
\hline
a^5+4a^4b+6a^3b^2+4a^2b^3+ab^4 \\
\qquad a^4b+4a^3b^2+6a^2b^3+4ab^4+b^5 \\
\hline
\end{array}$$

$$(a+b)^5=a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5.$$

3. Find the fifth and lower powers of $a-b$.

$$(a-b)^1 = a-b.$$
$$\begin{array}{l} a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline \end{array}$$
$$(a-b)^2 = a^2-2ab+b^2.$$
$$\begin{array}{l} a-b \\ \hline a^3-2a^2b+ab^2 \\ \quad -a^2b+2ab^2-b^3 \\ \hline \end{array}$$
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3.$$
$$\begin{array}{l} a-b \\ \hline a^4-3a^3b+3a^2b^2-ab^3 \\ \quad -a^3b+3a^2b^2-3ab^3+b^4 \\ \hline \end{array}$$
$$(a-b)^4 = a^4-4a^3b+6a^2b^2-4ab^3+b^4.$$
$$\begin{array}{l} a-b \\ \hline a^5-4a^4b+6a^3b^2-4a^2b^3+ab^4 \\ \quad -a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5 \\ \hline \end{array}$$
$$(a-b)^5 = a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5.$$

4. What is the cube of $a-x$?

Ans. $a^3-3a^2x+3ax^2-x^3$.

5. What is the square of $m+n-x$?

Ans. $m^2+2mn-2mx+n^2-2nx+x^2$.

6. What is the fourth power of $3x-2y$?

Ans. $81x^4-216x^3y+216x^2y^2-96xy^3+16y^4$.

7. What is the square of $a+b$?

Ans. $a^2+2ab+b^2$.

8. What is the square of $a+b+c$?

Ans. $a^2+2ab+2ac+b^2+2bc+c^2$.

EVOLUTION.

(95.) **Evolution** is the extracting of roots, or the reverse process of involution.

When the quantity whose root is to be found is a single letter, the operation is denoted by giving it a fractional exponent, the denominator of which denotes the degree of the root. (Art. 14.)

And in the same way we may denote the extraction of a root of any quantity or expression, by enclosing it within a parenthesis, and then treating it as a single letter.

Thus, the second root of $my = (my)^{\frac{1}{2}}$,

the third root of $x + y = (x + y)^{\frac{1}{3}}$,

the fourth root of $2x - 3y = (2x - 3y)^{\frac{1}{4}}$,

the nth root of $a - b = (a - b)^{\frac{1}{n}}$,

&c., &.,

CASE I.

(96.) To extract a root of a monomial, we obviously have the following

RULE.

I. *Extract the required root of the coefficient, by the usual arithmetical rule. When the root can not be accurately ob-*

tained, it may be denoted by means of a fractional exponent, the same as in the case of a letter.

II. *Extract the required root of the different letters, by multiplying the exponents which they already have by the fractional exponent denoting the required root. To this root prefix the root of the coefficient.*

Note.—Since the *even* powers of all quantities, whether positive or negative, are positive; it follows that an *even* root of a negative quantity is *impossible*, and an *even* root of a positive quantity is either positive or negative.

We also infer that an *odd* root of any quantity has the same sign as the quantity itself.

EXAMPLES.

1. What is the square root of $64a^2b^4x^6$?

In this example, the square root of the coefficient, 64, is ± 8, where we have used both signs.

And,

$$(a^2b^4x^6)^{\frac{1}{2}} = ab^2x^3,$$

$$\therefore\ (64a^2b^4x^6)^{\frac{1}{2}} = \pm 8ab^2x^3.$$

2. What is the cube root of $64a^3x^6$?

Ans. $4ax^2$.

3. What is the fifth root of $-32xy^2$?

Ans. $-2x^{\frac{1}{5}}y^{\frac{2}{5}}$.

4. What is the seventh root of $-ax^{-2}$?

Ans. $-a^{\frac{1}{7}}x^{-\frac{2}{7}} = -\dfrac{a^{\frac{1}{7}}}{x^{\frac{2}{7}}}$.

5. What is the square root of $-4a^4b^2$?

Ans. Impossible.

6. What is the cube root of $27a^3b^{12}$?

Ans. $3ab^4$.

7. What is the fourth root of $16a^{-3}bx^{-1}$?

$$\text{Ans. } \pm 2a^{-\frac{3}{4}}b^{\frac{1}{4}}x^{-\frac{1}{4}} = \pm\frac{2b^{\frac{1}{4}}}{a^{\frac{3}{4}}x^{\frac{1}{4}}}.$$

(97.) By comparing the operations of this rule, with those of rule under Art. 93, we see that involution and evolution of monomials may both be performed by one general rule, of multiplying the exponents of the respective letters by the exponent denoting the power or root. We will therefore give the following promiscuous examples, which will require the aid of one or both of these rules.

EXAMPLES.

1. What is the cube root of the second power of $8a^3b^9$?

If we first raise $8a^3b^9$ to the second power, it will become

$$(8a^3b^9)^2 = 64a^6b^{18},$$

extracting the third root, we find

$$(64a^6b^{18})^{\frac{1}{3}} = 4a^2b^6,$$

for the result required.

Again, first extracting the cube root of $8a^3b^9$, it becomes

$$(8a^3b^9)^{\frac{1}{3}} = 2ab^3,$$

raising this to the second power, it becomes

$$(2ab^3)^2 = 4a^2b^6,$$

the same as before.

(98.) Hence, the cube root of the square of a quantity, is the same as the square of the cube root of the same quantity.

And in general, *the nth root of the mth power of a quantity, is the same as the mth power of the nth root of the same quantity.*

Therefore, $a^{\frac{4}{5}}$ may be read, the fourth power of the fifth root of a, or the fifth root of the fourth power of a.

And in the same way, $(a+b)^{\frac{3}{2}}$ is read, the third power of the square root of the sum of a and b, or the square root of third power of the sum of a and b.

2. What is the value of $(-3ab^2x^3)^{\frac{2}{3}}$?

Ans. $3^{\frac{2}{3}}a^{\frac{2}{3}}b^{\frac{4}{3}}x^2$.

3. What is the value of $(4a^{-2}b^4x)^{\frac{5}{2}}$?

Ans. $\pm 32a^{-5}b^{10}x^{\frac{5}{2}}$.

(99.) Surd quantities may be made to assume several equivalent forms which require to be read differently. As an example, the surd $a^{-\frac{3}{5}}$ may be written six different ways, as follows:

1. $\left((a^3)^{\frac{1}{5}}\right)^{-1}$; 2. $\left((a^{\frac{1}{5}})^3\right)^{-1}$; 3. $\left((a^{-1})^{\frac{1}{5}}\right)^3$;

4. $\left((a^{\frac{1}{5}})^{-1}\right)^3$; 5. $\left((a^{-1})^3\right)^{\frac{1}{5}}$; 6. $\left((a^3)^{-1}\right)^{\frac{1}{5}}$.

These six expressions are read as follows:

1. The reciprocal of the fifth root of the third power of a.

2. The reciprocal of the third power of the fifth root of a.

3. The third power of the fifth root of the reciprocal of a.

4. The third power of the reciprocal of the fifth root of a.

5. The fifth root of the third power of the reciprocal of a.

6. The fifth root of the reciprocal of the third power of a.

CASE I.

(100.) To extract any root of a polynomial, we have the following general

RULE.

I. *Having arranged the polynomial according to the powers of some one of the letters, so that the highest power shall stand first, extract the required root of the first term, which will be the first term of the root sought.*

II. *Subtract the power of this first term of the root from the polynomial, and divide the first term of the remainder, by the first term of the root involved to the next inferior power, multiplied by the number denoting the root; the quotient will be the second term of the root.*

III. *Subtract the power of the terms already found from the polynomial, and using the same divisor proceed as before.*

This rule obviously verifies itself, since, whenever a new term is added to the root, the whole is raised to the given power, and the result is subtracted from the given polynomial: and when we thus find a power equal to the given polynomial, it is evident that the true root has been found.

1. What is the fifth root of

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\,?$$

OPERATION.

ROOT.

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\,(a + b.$$
$$a^5$$
$$5a^4)\quad 5a^4b$$
$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5.$$
$$0$$

EXPLANATION.

We first found the fifth root of the first term a^5, to be a, which we placed to the right of the polynomial for the first term of the root. Raising a to the fifth power and subtracting it from the polynomial, we have $5a^4b$ for the first term of the remainder.

Since the number denoting the root is 5, we raise the first term of the root, a, to the fourth power, which thus becomes a^4, this multiplied by the number denoting the root, gives $5a^4$ for our divisor.

Now, dividing $5a^4b$ by $5a^4$, we get b, which we write for the second term of the root.

Involving this root to the fifth power by actual multiplication, as was done in Ex. 2, Art. 94, we have

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5\,;$$

which subtracted from the given polynomial, leaves no remainder, so that we know that $a + b$ is the true root.

2. What is the square root of

$$4x^4 - 16x^3 + 24x^2 - 16x + 4\,?$$

OPERATION.

$$\begin{array}{rl}
 & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{ROOT.} \\
 & 4x^4 - 16x^3 + 24x^2 - 16x + 4\,(2x^2 - 4x + 2 \\
(2x^2)^2 = & \underline{4x^4} \\
 & 4x^2)-16x^3 \\
(2x^2-4x)^2 = & \underline{4x^4 - 16x^3 + 16x^2} \\
 & \qquad\qquad\quad 4x^2)8x^2 \\
(2x^2-4x+2)^2 = & \underline{4x^4 - 16x^3 + 24x^2 - 16x + 4} \\
 & \qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}$$

3. What is the square root of

$$16x^4+24x^3+89x^2+60x+100\ ?$$

Ans. $4x^2+3x+10$.

4. What is the cube root of

$$a^6+3a^5-3a^4-11a^3+6a^2+12a-8\ ?$$

Ans. a^2+a-2.

5. What is the sixth root of

$$a^6-6a^5b+15a^4b^2-20a^3b^3+15a^2b^4-6ab^5+b^6\ ?$$

Ans. $a-b$.

6. What is the fourth root of

$$a^4-4a^3b+6a^2b^2-4ab^3+b^4\ ?$$

Ans. $a-b$.

(101.) If we carefully observe the law by which a polynomial is raised to the second power, we shall, by reversing the process, be enabled to deduce a rule for the extraction of the square root of a polynomial, which will be more simple than the above general rule, and of more interest, since the arithmetical rule is deduced from it.

By actual multiplication, we find

$$(a+b)^2=a^2+2ab+b^2,$$

$$(a+b+c)^2=a^2+2ab+b^2+2(a+b)c+c^2,$$

$$(a+b+c+d)^2$$
$$=a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2,$$

$$(a+b+c+d+e)^2$$
$$=\left\{\begin{array}{l}a^2+2ab+b^2+2(a+b)c+c^2\\+2(a+b+c)d+d^2+2(a+b+c+d)e+e^2.\end{array}\right\}$$

&c. &c.

From the above, we discover, that

(102.) *The square of any polynomial is equal to the square of the first term, plus twice the first term into the second, plus the square of the second; plus twice the sum of the first two into the third, plus the square of the third; plus twice*

the sum of the first three into the fourth, plus the square of the fourth; and so on.

(103.) Hence, the square root of a polynomial can be found by the following

RULE.

I. ***After arranging the polynomial according to the powers of some one of the letters, take the root of the first term for the first term of the required root, and subtract its square from the polynomial.***

II. ***Bring down the next two terms for a dividend. Divide its first term by twice the root just found, and add the quotient, both to the root, and to the divisor. Multiply the divisor, thus increased, into the term last placed in the root, and subtract the product from the dividend.***

III. ***Bring down two or three additional terms, and proceed as before.***

EXAMPLES.

1. What is the square root of

$$a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2\,?$$

OPERATION.

$$\begin{array}{lll}
 & a^2+2ab+b^2+2(a+b)c+c^2+2(a+b+c)d+d^2 & \left[\;\overset{\text{ROOT.}}{a+b+c+d.}\right. \\
 & a^2 & \\
\hline
 & 2ab+b^2 & \\
 & 2ab+b^2 & \\
\hline
2(a+b)+c & 2(a+b)c+c^2 & \\
 & 2(a+b)c+c^2 & \\
\hline
2(a+b+c)+d & 2(a+b+c)d+d^2 & \\
 & 2(a+b+c)d+d^2 & \\
\hline
 & 0 &
\end{array}$$

2. What is the square root of

$$4x^6+12x^5+5x^4-2x^3+7x^2-2x+1\ ?$$

OPERATION.

$$\begin{array}{lll}
 & 4x^6+12x^5+5x^4-2x^3+7x^2-2x+1 & (2x^3+3x^2-x+1. \quad \text{ROOT.} \\
 & 4x^6 & \\
\hline
4x^3+3x^2 & 12x^5+5x^4 & \\
 & 12x^5+9x^4 & \\
\hline
4x^3+6x^2-x & -4x^4-2x^3+7x^2 & \\
 & -4x^4-6x^3+\ x^2 & \\
\hline
4x^3+6x^2-2x+1 & 4x^3+6x^2-2x+1 & \\
 & 4x^3+6x^2-2x+1 & \\
\hline
 & 0 &
\end{array}$$

3. What is the square root of

$$x^4-2x^2y^2-2x^2+y^4+2y^2+1\ ?$$

Ans. x^2-y^2-1.

4. What is the square root of

$$9x^4y^4-30x^3y^3+25x^2y^2\ ?$$

Ans. $3x^2y^2-5xy$.

5. What is the square root of

$$a^2+2ab-2ac+b^2-2bc+c^2\ ?$$

Ans. $a+b-c$.

6. What is the square root of

$$4m^2-36mn+81n^2\ ?$$

Ans. $2m-9n$.

In these examples, and in all others where an *even* root is extracted, the terms of the root may have all their signs changed, and still satisfy the questions.

(104.) We will now endeavor to find a particular rule for the extraction of the cube root of a polynomial.

By actual multiplication, we find

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3,$$

$$(a+b+c)^3$$
$$=a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2+c^3,$$

$$(a+b+c+d)^3$$
$$=\left\{\begin{array}{l} a^3+3a^2b+3ab^2+b^3+3(a+b)^2c+3(a+b)c^2 \\ +c^3+3(a+b+c)^2d+3(a+b+c)d^2+d^3. \end{array}\right\}$$

&c., &c.

(105.) From which we discover that

The cube of any polynomial is equal to the cube of the first term, plus three times the square of the first into the second, plus three times the first into the square of the second, plus the cube of the second; plus three times the square of the sum of the first two into the third, plus three times the sum of the first two into the square of the third, plus the cube of the third; plus three times the square of the sum of the first three into the fourth, plus three times the sum of the first three into the square of the fourth, plus the cube of the fourth; and so on.

(106.) Now we may reverse the above process, that is, we may extract the cube root of a polynomial by the following

RULE.

I. *Having arranged the terms of the polynomial according to the powers of some one of the letters, seek the cube root of the first term, which place at the right of the polynomial for the first term of the root, also place it at the left by itself, for the first term of a column, headed,* FIRST

COLUMN. *Then multiply it into itself, and place the product for the first term of a column, headed,* SECOND COLUMN. *Again, multiply this last result, by the same first term of the root and subtract the product from the first term of the polynomial, and then bring down the next three terms of the polynomial, for the* FIRST DIVIDEND. *Add the first term of the root just found to the first term of the first column, the sum will constitute its second term, which must be multiplied by the first term of the root, and the result added to the first term of the second column, for its second term, which we will call the* FIRST TRIAL DIVISOR. *The same first term of the root must be added to the second term of the first column, forming its third term.*

II. *Divide the first term of the first dividend by the first term of the trial divisor, the quotient must be added to the root already found, for its second term, it must also be added to the last term of the first column, the result will be its fourth term, which must be multiplied by the second term of the root, and the product added to the last term of the second column, which sum will give its third term, which in turn must be multiplied by the second term of the root, and the product subtracted from the first dividend.*

III. *To the remainder bring down three or four of the next terms of the polynomial for a* SECOND DIVIDEND. *Proceed with this second term of the root, precisely as was done with the first term, and so continue until the entire polynomial has been exhausted.*

EXAMPLES.

1. What is the cube root of $a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$?

OPERATION.

FIRST COLUMN.	SECOND COLUMN.		ROOT.
		$a^3 + 3a^2b + 3ab^2 + b^3 + 3(a+b)^2c + 3(a+b)c^2 + c^3$	$[a+b+c.$
a	a^2	a^3	
$2a$	$3a^2$		
$3a$	$3a^2 + 3ab + b^2$	$3a^2b + 3ab^2 + b^3$	
$3a+b$	$\left\{ \begin{array}{l} 3a^2 + 6ab + 3b^2 \\ \text{or } 3(a+b)^2 \end{array} \right.$	$3a^2b + 3ab^2 + b^3$	
$3a+2b$			
$\left\{ \begin{array}{l} 3a+3b \\ \text{or } 3(a+b) \end{array} \right.$	$3(a+b)^2 + 3(a+b)c + c^2$	$3(a+b)^2c + 3(a+b)c^2 + c^3$	
		$3(a+b)^2c + 3(a+b)c^2 + c^3$	
$3(a+b)+c$		0	

2. What is the cube root of $8x^3 + 36x^2 + 54x + 27$?

OPERATION.

FIRST COLUMN.	SECOND COLUMN.		ROOT.
		$8x^3 + 36x^2 + 54x + 27$	$(2x+3.$
$2x$	$4x^2$	$8x^3$	
$4x$	$12x^2$		
$6x+3$	$12x^2 + 18x + 9$	$36x^2 + 54x + 27$	
		$36x^2 + 54x + 27$	

3. What is the cube root of the polynomial $27x^6-54x^5+63x^4-44x^3+21x^2-6x+1$?

FIRST COLUMN.	SECOND COLUMN.	
		$27x^6-54x^5+63x^4-44x^3+21x^2-6x+1(3x^2-2x+1$
$3x^2$	$9x^4$	$27x^6$
$6x^2$	$27x^4$	
$9x^2$	$27x^4-18x^3+4x^2$	$-54x^5+63x^4-44x^3$
$9x^2-2x$	$27x^4-36x^3+12x^2$	$-54x^5+36x^4-8x^3$
$9x^2-4x$	$27x^4-36x^3+21x^2-6x+1$	$27x^4-36x^3+21x^2-6x+1$
$9x^2-6x$		$27x^4-36x^3+21x^2-6x+1$
$9x^2-6x+1$		0

4. What is the cube root of the polynomial $x^6+6x^5-40x^3+96x-64$?

FIRST COLUMN.	SECOND COLUMN.		
			$x^6+6x^5-40x^3+96x-64(x^2+2x-4.$
x^2	x^4		x^6
$2x^2$	$3x^4$		$6x^5 \quad -40x^3$
$3x^2$	$3x^4+6x^3+4x^2$		$6x^5+12x^4+8x^3$
$3x^2+2x$	$3x^4+12x^3+12x^2$		
$3x^2+4x$	$3x^4+12x^3$	$-24x+16$	$-12x^4-48x^3+96x-64$
$3x^2+6x$			$-12x^4-48x^3+96x-64$
$3x^2+6x-4$			0

5. What is the cube root of the polynomial

$$x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1\ ?$$

Ans. $x^2 - 2x + 1$.

6. What is the cube root of the polynomial

$$a^9 + 12a^5x^2 - 8a^3x^3 - 6a^7x\ ?$$

Ans. $a^3 - 2ax$.

7. What is the cube root of

$$a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1\ ?$$

Ans. $a^2 - a + 1$.

8. What is the cube root of

$$x^6 + 6x^5 + 21x^4 + 44x^3 + 63x^2 + 54x + 27\ ?$$

Ans. $x^2 + 2x + 3$.

(107.) From the above rule, for extracting the cube root of a polynomial, we can easily deduce the rule which we have given in the Higher Arithmetic for the extraction of the cube root of a number.

This rule is also particularly interesting because of its close analogy to the method of finding the numerical roots of a cubic equation, as explained in a subsequent part of this work.

IRRATIONAL OR SURD QUANTITIES.

(108.) AN IRRATIONAL QUANTITY, OR SURD, is a quantity affected with a fractional exponent or radical, without which, it can not be accurately expressed.

Thus,

$\sqrt{3}$ is a surd, since the square root of 3 can not be accurately found; also $8^{\frac{1}{2}}$, $4^{\frac{1}{3}}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$, &c., are surd quantities.

REDUCTION OF SURDS.

CASE I.

(109.) To reduce a rational quantity to the form of a surd, we have this

RULE.

Raise the quantity to a power denoted by the root of the required surd; then the corresponding root of this power, expressed by means of a radical sign or fractional exponent, will express the quantity under the proposed form.

EXAMPLES.

1. Reduce $5a$ to the form of the cube root.

Raising $5a$ to the third power, we have

$$(5a)^3 = 125a^3;$$

extracting the cube root, it becomes

$$5a = \sqrt[3]{125a^3} = (125a^3)^{\frac{1}{3}}.$$

2. Reduce $\frac{x^3}{a^3}$ to the form of the fifth root.

$$\text{Ans. } \frac{x^3}{a^3} = \sqrt[5]{\frac{x^{15}}{a^{15}}} = \left(\frac{x^{15}}{a^{15}}\right)^{\frac{1}{5}}.$$

3. Reduce $\frac{\sqrt{a}}{y}$ to the form of the fourth root.

$$\text{Ans. } \frac{\sqrt{a}}{y} = \left(\frac{a^2}{y^4}\right)^{\frac{1}{4}}.$$

4. Reduce $\frac{a^2}{b^3}$ to the form of the nth root.

$$\text{Ans. } \frac{a^2}{b^3} = \left(\frac{a^{2n}}{b^{3n}}\right)^{\frac{1}{n}}.$$

CASE II.

(110.) To reduce surds expressing different roots to equivalent ones expressing the same root.

Reduce the different indices to common denominators; then raise each quantity to a power denoted by the numerator of its respective exponent; afterwards take the root denoted by the common denominator.

EXAMPLES.

1. Reduce $\sqrt{3}$, $\sqrt[3]{4}$, and $\sqrt[4]{5}$ to surds expressing the same root.

Changing the radicals into fractional exponents, they become $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, which reduced to a common denominator, are

$\frac{6}{12}, \frac{4}{12}, \frac{3}{12}$. Now, raising the quantities 3, 4, and 5, to powers denoted respectively by 6, 4, and 3, we find 3^6, 4^4, 5^3, or, which is the same, 729, 256, 125. Taking the 12th root of these results, they become

$$(729)^{\frac{1}{12}}, (256)^{\frac{1}{12}}, (125)^{\frac{1}{12}}.$$

2. Reduce $a^{\frac{1}{2}}$ and $x^{\frac{2}{3}}$ to surds expressing the same root.

$$\text{Ans.} \begin{cases} a^{\frac{1}{2}} = (a^3)^{\frac{1}{6}}, \\ x^{\frac{2}{3}} = (x^4)^{\frac{1}{6}}. \end{cases}$$

3. Reduce $x^{\frac{1}{3}}, y^{\frac{2}{3}}, m^{\frac{1}{2}}$ to surds expressing the same root

$$\text{Ans.} \begin{cases} x^{\frac{1}{3}} = (x^2)^{\frac{1}{6}}, \\ y^{\frac{2}{3}} = (y^4)^{\frac{1}{6}}, \\ m^{\frac{1}{2}} = (m^3)^{\frac{1}{6}}. \end{cases}$$

4. Reduce $\sqrt[3]{2}$, $\sqrt[4]{3}$, $\sqrt[5]{4}$ to surds expressing the same root.

$$\text{Ans.} \begin{cases} \sqrt[3]{2} = (2^{20})^{\frac{1}{60}}, \\ \sqrt[4]{3} = (3^{15})^{\frac{1}{60}}, \\ \sqrt[5]{4} = (4^{12})^{\frac{1}{60}}. \end{cases}$$

CASE III.

(111.) To reduce surds to their simplest form. When ever a surd can be separated into two factors, one of which is a perfect power, it can be simplified by this

RULE.

Having separated the surd into two factors, one of which is a perfect power, take the root of the factor which is a perfect power, and multiply it by the surd of the other factor.

EXAMPLES.

1. Reduce $\sqrt{288}$ to its simplest form.

We can separate 288 into the factors 144×2, of which 144 is a perfect square whose root is 12; therefore

$$\sqrt{288} = \sqrt{144 \times 2} = \sqrt{144} \times \sqrt{2} = 12\sqrt{2}.$$

2. Reduce $\sqrt[3]{x^3y - a^2x^3}$ to its simplest form.

Ans. $\sqrt[3]{x^3y - a^2x^3} = x\sqrt[3]{y - a^2}$.

3. Reduce $\sqrt[5]{-32a^5b}$ to its simplest form.

Ans. $\sqrt[5]{-32a^5b} = -2a\sqrt[5]{b}$.

4. Reduce $(a^2x^6y^{-4})^{\frac{1}{2}}$ to its simplest form.

Ans. $(a^2x^6y^{-4})^{\frac{1}{2}} = ax^3y^{-2} = \dfrac{ax^3}{y^2}$.

5. Reduce $(m^2nx^5y^3)^{\frac{1}{2}}$ to its simplest form.

Ans. $mx^2y(nxy)^{\frac{1}{2}}$.

(112.) When a surd is in the form of a fraction, it may be simplified by the following

RULE.

Multiply both numerator and denominator by such a quantity as will render the denominator a perfect power.

EXAMPLES.

1. Reduce $\sqrt{\frac{8}{11}}$ to its simplest form.

Multiplying both numerator and denominator by 11, we have

$$\sqrt{\frac{8}{11}}=\sqrt{\frac{88}{121}}=\sqrt{\frac{4}{121}\times 22}=\frac{2}{11}\sqrt{22}.$$

2. Reduce $\sqrt[3]{\frac{ab^2}{x^2}}$ to its simplest form.

$$\text{Ans. } \sqrt[3]{\frac{ab^2}{x^2}}=\sqrt[3]{\frac{ab^2x}{x^3}}=\frac{1}{x}\sqrt[3]{ab^2x}.$$

3. Reduce $\left(\frac{a^4b^8}{xy}\right)^{\frac{1}{5}}$ to its simplest form.

$$\text{Ans. } \left(\frac{a^4b^8}{xy}\right)^{\frac{1}{5}}=\frac{b}{xy}\,(a^4b^3x^4y^4)^{\frac{1}{5}}.$$

4. Reduce $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{3}}$ to its simplest form

$$\text{Ans. } \left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{3}}=\left(\frac{1}{ab^2x}\right)^{\frac{1}{3}}=\frac{1}{abx}(a^2bx^2)^{\frac{1}{3}}.$$

ADDITION AND SUBTRACTION OF SURDS.

RULE.

(113.) *Reduce the surds to their simplest form; then, if the surd part is the same in both, add or subtract the rational parts, and annex the common surd part to the result; but when the surd parts are different, they can only be added or subtracted by the aid of the signs + or −.*

EXAMPLES.

1. What is the sum of $\sqrt{54}$ and $\sqrt{24}$? Also, what is the difference of the same surds?

By reduction we have

$$\sqrt{54}=\sqrt{9\times 6}=3\sqrt{6}$$
$$\sqrt{54}=\sqrt{4\times 6}=2\sqrt{6}$$

Therefore, $\sqrt{54}+\sqrt{24} \quad =5\sqrt{6}.$

And, $\sqrt{54}-\sqrt{24} \quad = \sqrt{6}.$

2. What is the sum and difference of $\sqrt[3]{a^4b}$ and $\sqrt[3]{ab^4}$?

Ans. $\begin{cases} \text{The sum } =(a+b)\sqrt[3]{ab}. \\ \text{The diff. } =(a-b)\sqrt[3]{ab}. \end{cases}$

3. What is the sum of $(36x^2y)^{\frac{1}{2}}$ and $(25y)^{\frac{1}{2}}$?

Ans. $(6x+5)\sqrt{y}$.

4. What is the sum of $(8x)^{\frac{1}{3}}$, $(xy^6)^{\frac{1}{3}}$, and $(27x^4)^{\frac{1}{3}}$?

Ans. $(2+3x+y^2)\sqrt[3]{x}$.

5. What is the sum of $(ab^2x^6)^{\frac{1}{5}}$ and $(m^4y^{10})^{\frac{1}{5}}$?

Ans. $x(ab^2x)^{\frac{1}{5}}+y^2(m^4)^{\frac{1}{5}}$.

MULTIPLICATION AND DIVISION OF SURDS.

RULE.

(114.) *Reduce the surds to equivalent ones expressing the same root*, (Case II. Art. 110,) *then multiply or divide as required.*

EXAMPLES.

1. What is the product of $\sqrt{8}$ by $\sqrt[3]{16}$?

By Case II, we find $\sqrt{8}=(8^3)^{\frac{1}{6}}=(512)^{\frac{1}{6}}$.

$$\sqrt[3]{16}=(16^2)^{\frac{1}{6}}=(256)^{\frac{1}{6}}.$$

Therefore, $\sqrt{8}\times\sqrt[3]{16}=(512\times 256)^{\frac{1}{6}}=4\sqrt[6]{32}$

2. What is the product of $4\sqrt[3]{ab}$ by $3\sqrt{by}$?

Ans. $12\sqrt[6]{a^2b^5y^3}$.

3. Divide $4\sqrt[6]{32}$ by $\sqrt[3]{16}$.

Ans. $\sqrt{8}$.

4. Divide $\sqrt{a^2b^{-3}}$ by $\sqrt[3]{ab^4}$.

Ans. $ab^{-2}(a^{-2}b^{-5})^{\frac{1}{6}}=\frac{a}{b^2}\left(\frac{1}{a^2b^5}\right)^{\frac{1}{6}}$.

5. Divide $\sqrt{4a^3b^3}$ by $\sqrt[3]{2ab^2}$.

Ans. $a(16ab^5)^{\frac{1}{6}}$.

EXTRACTION OF THE SQUARE ROOT OF A BINOMIAL SURD.

(115.) When one or both of the terms of a binomial are surds, it is called a *binomial surd*.

Thus,

$$\left.\begin{array}{r} a\pm\sqrt{b} \\ \sqrt{c}\pm\sqrt{d} \end{array}\right\} \text{ are binomial surds.}$$

(116.) Before we proceed to the extraction of the square root of a binomial surd, we will establish the following *lemmas*.

LEMMA I.

The square root of a rational quantity can not consist of the sum of two parts, one of which is rational and the other irrational.

For if possible, suppose we have the relation

$$\sqrt{a}=x+\sqrt{y}, \tag{1}$$

where x is rational.

Squaring both members of (1), we find

$$a=x^2+2x\sqrt{y}+y \tag{2}$$

From (2) we obtain

$$\sqrt{y}=\frac{a-x^2-y}{2x}. \tag{3}$$

Equation (3) gives an irrational quantity equal to a rational one, which is impossible, therefore the condition (1) is impossible, hence the above lemma must be correct.

LEMMA II.

In any equation, consisting of rational quantities and irrational quantities, the rational quantities on each side are equal, as also are the irrational quantities.

Suppose we have the equation

$$a + \sqrt{b} = x + \sqrt{y}. \qquad (1)$$

Then if a is not equal to x, let us have $a = x \pm m$, this value of a substituted in (1), gives

$$x \pm m + \sqrt{b} = x + \sqrt{y}, \qquad (2)$$

or

$$\pm m + \sqrt{b} = \sqrt{y}. \qquad (3)$$

Equation (3) shows that the square root of y is partly rational and partly irrational, which is impossible (Lemma I). Therefore it is absurb to suppose that x differs in value from a, hence $x = a$. Consequently $\sqrt{b} = \sqrt{y}$. So that the above lemma is correct.

LEMMA III.

If $\sqrt{a + \sqrt{b}} = x + \sqrt{y}$, *then will* $\sqrt{a - \sqrt{b}} = x - \sqrt{y}$.

If we square both the members of the equation.

$$\sqrt{a + \sqrt{b}} = x + \sqrt{y}, \qquad (1)$$

we find

$$a + \sqrt{b} = x^2 + 2x\sqrt{y} + y. \qquad (2)$$

Equating the rational, as well as the irrational parts of (2), (Lemma II), we have

$$a = x^2 + y, \qquad (3)$$

$$\sqrt{b} = 2x\sqrt{y}. \qquad (4)$$

Subtracting (4) from (3), we have

$$a-\sqrt{b}=x^2-2x\sqrt{y}+y. \qquad (5)$$

Extracting the square root of both members of (5), we get

$$\sqrt{a-\sqrt{b}}=x-\sqrt{y}. \qquad (6)$$

So that if (1) is true, then also will (6) be true, which establishes the above lemma.

(117.) We are now prepared to proceed to the extraction of a binomial surd.

Assume

$$\sqrt{a+\sqrt{b}}=x+\sqrt{y}. \qquad (1)$$

Then, (Lemma III)

$$\sqrt{a-\sqrt{b}}=x-\sqrt{y}. \qquad (2)$$

Equations (1) and (2), when squared, become

$$a+\sqrt{b}=x^2+2x\sqrt{y}+y. \qquad (3)$$
$$a-\sqrt{b}=x^2-2x\sqrt{y}+y. \qquad (4)$$

Taking the sum of (3) and (4), and dividing the result by 2, we obtain

$$a=x^2+y. \qquad (5)$$

If we multiply together equations (1) and (2), we get

$$\sqrt{a^2-b}=x^2-y. \qquad (6)$$

Adding (5) and (6), and dividing by 2, we have

$$\frac{a+\sqrt{a^2-b}}{2}=x^2. \qquad (7)$$

Subtracting (6) from (5), and dividing by 2, we find

$$\frac{a-\sqrt{a^2-b}}{2}=y. \qquad (8)$$

Extracting the square root of both members of (7) and (8), we get

$$x=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \qquad (9)$$

$$\sqrt{y}=\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \qquad (10)$$

Taking the sum of (9) and (10), we have

$$x+\sqrt{y}=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}+\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \quad (11)$$

Subtracting (10) from (9), we get

$$x-\sqrt{y}=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}-\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \quad (12)$$

For the left-hand members of (11) and (12), substitute their values given by (1) and (2), and we then have

$$\sqrt{a+\sqrt{b}}=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}+\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \ (A)$$

$$\sqrt{a-\sqrt{b}}=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}-\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \ (B)$$

By using the double sign $\pm$, we may combine in one formula, both (A) and (B).

$$\sqrt{a\pm\sqrt{b}}=\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}\pm\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}. \ (C)$$

(118.) We will now show the use of formulas (A) and (B) by the following

EXAMPLES.

1. What is the square root of $7-2\sqrt{10}$?

Reducing the factor 2, to the form of the square root (Art. 109), and then introducing it under the radical sign, we have $7-2\sqrt{10}=7-\sqrt{40}$, which, when compared with the general form $a-\sqrt{b}$, gives $a=7$; $b=40$, these values of a and b, substituted in (B), give

$$\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{7+\sqrt{49-40}}{2}\right\}^{\frac{1}{2}}=\left(\frac{7+3}{2}\right)^{\frac{1}{2}}=\sqrt{5}$$

$$\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{7-\sqrt{49-40}}{2}\right\}^{\frac{1}{2}}=\left(\frac{7-3}{2}\right)^{\frac{1}{2}}=\sqrt{2}.$$

Therefore, we have

$$\sqrt{7-2\sqrt{10}}=\sqrt{5}-\sqrt{2}.$$

2. What is the square root of $6+\sqrt{20}$?

In this example we have $a=6$; $b=20$, which substituted in formula (A), gives

$$\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{6+\sqrt{36-20}}{2}\right\}^{\frac{1}{2}}=\left(\frac{6+4}{2}\right)^{\frac{1}{2}}=\sqrt{5}.$$

$$\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{6-\sqrt{36-20}}{2}\right\}^{\frac{1}{2}}=\left(\frac{6-4}{2}\right)^{\frac{1}{2}}=1.$$

Therefore,

$$\sqrt{6+\sqrt{20}}=\sqrt{5}+1.$$

3. What is the square root of $2(x+1)+4\sqrt{x}$?

Ans. $\sqrt{2x}+\sqrt{2}$.

4. What is the square root of $6 - 2\sqrt{5}$?

Ans. $\sqrt{5} - 1$.

5. What is the square root of $7 + 4\sqrt{3}$?

Ans. $2 + \sqrt{3}$.

TO FIND MULTIPLIERS WHICH WILL CAUSE SURDS TO BECOME RATIONAL.

CASE I.

(119.) When the surd consists of but one term, we can proceed as follows:

Suppose the given surd is $x^{\frac{1}{m}}$, if we multiply this by $x^{\frac{m-1}{m}}$, by rule under Art. 114, we shall have $x^{\frac{1}{m}} \times x^{\frac{m-1}{m}} = x$, a rational quantity.

Hence, to cause a monomial surd to become rational by multiplication, we have this

RULE.

Multiply the surd by the same quantity, having such an exponent, as when added to the exponent of the given surd, shall make a unit.

EXAMPLES.

1. How can the surd $x^{\frac{1}{3}}$ be made rational by multiplication.

In this example, $\frac{2}{3}$ added to the exponent $\frac{1}{3}$, gives 1, therefore we must multiply by $x^{\frac{2}{3}}$, performing the operation, we have

$$x^{\frac{1}{3}} \times x^{\frac{2}{3}} = x.$$

2. Multiply $x^{\frac{3}{5}}$ so that it shall become rational.

Ans. $x^{\frac{3}{5}} \times x^{\frac{2}{5}} = x$.

3. Multiply $x^{\frac{4}{7}}$ so that it shall become rational.

Ans. $x^{-\frac{4}{7}} \times x^{\frac{11}{7}} = x$.

CASE II.

(120.) When the surd consists of two terms, or is a binomial surd.

Suppose it is required to multiply $\sqrt{a}+\sqrt{b}$ so as to produce a rational product; we know from Art. 35, Theorem III, that

$$(\sqrt{a}+\sqrt{b})\times(\sqrt{a}-\sqrt{b})=a-b.$$

Hence, to cause a binomial surd to become rational by multiplication, we have this

RULE.

Change the sign which connects the two terms of the binomial surd, from + to —, or from — to +, and this result, multiplied by the binomial surd, will give a rational product.

EXAMPLES.

1. Multiply $\sqrt{3}-\sqrt{2}$ so as to obtain a rational product.

Ans. $(\sqrt{3}-\sqrt{2})\times(\sqrt{3}+\sqrt{2})=3-2=1$.

2. Multiply $4+\sqrt{5}$ so that the result shall be rational.

Ans. $(4+\sqrt{5})\times(4-\sqrt{5})=11$.

3. How can $\sqrt{a+b}-\sqrt{a-b}$ be made rational by multiplication?

Ans. $(\sqrt{a+b}-\sqrt{a-b})\times(\sqrt{a+b}+\sqrt{a-b})=2b$.

4. How can $\sqrt{7}-1$ become rational by multiplication?

Ans. $(\sqrt{7}-1)\times(\sqrt{7}+1)=6$.

(121.) If the surd consist of three or more terms of the square root, connected by the signs plus and minus, it can be made rational, by first multiplying it by itself after changing one or more of the connecting signs.

EXAMPLES.

1. If it is required to make $\sqrt{5}-\sqrt{3}+\sqrt{2}$ rational by multiplication, we should first multiply by $\sqrt{5}+\sqrt{3}+\sqrt{2}$, by which means we obtain

$$\begin{array}{l}
\sqrt{5}-\sqrt{3}+\sqrt{2}\\
\sqrt{5}+\sqrt{3}+\sqrt{2}\\
\hline
5-\sqrt{15}+\sqrt{10}-3+\sqrt{6}\\
\quad+\sqrt{15}+\sqrt{10}\quad-\sqrt{6}+2\\
\hline
5\quad+2\sqrt{10}-3\quad+2=2\sqrt{10}+4.
\end{array}$$

Again, multiplying $2\sqrt{10}+4$ by $2\sqrt{10}-4$, we get

$$(2\sqrt{10}+4)\times(2\sqrt{10}-4)=24.$$

2. Multiply $2+\sqrt{3}-\sqrt{2}$ so that it shall become rational.

FIRST OPERATION.

$$\begin{array}{l}
2+\sqrt{3}-\sqrt{2}\\
2+\sqrt{3}+\sqrt{2}\\
\hline
4+2\sqrt{3}-2\sqrt{2}+3-\sqrt{6}\\
\quad+2\sqrt{3}+2\sqrt{2}\quad+\sqrt{6}-2\\
\hline
4+4\sqrt{3}\quad+3\quad-2=4\sqrt{3}+5.
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l}
4\sqrt{3}+5\\
4\sqrt{3}-5\\
\hline
48\quad+20\sqrt{3}\\
\quad-20\sqrt{3}-25\\
\hline
48\quad-25=23.
\end{array}$$

3. Multiply $\sqrt{5}+\sqrt{2}-\sqrt{3}+1$ so that its product shall be rational.

FIRST OPERATION.

$$\begin{array}{l}
\sqrt{5}+\sqrt{2}-\sqrt{3}+1 \\
\sqrt{5}-\sqrt{2}+\sqrt{3}+1 \\
\hline
5+\sqrt{10}-\sqrt{15}+\sqrt{5}+\sqrt{6}-\sqrt{2}+\sqrt{3} \\
-2-\sqrt{10}+\sqrt{15}+\sqrt{5}+\sqrt{6}+\sqrt{2}-\sqrt{3} \\
3 \\
1 \\
\hline
1 \qquad\qquad +2\sqrt{5}+2\sqrt{6}.
\end{array}$$

SECOND OPERATION.

$$\begin{array}{l}
1+2\sqrt{5}+2\sqrt{6} \\
1-2\sqrt{5}+2\sqrt{6} \\
\hline
1+2\sqrt{5}+2\sqrt{6}-4\sqrt{30} \\
-20-2\sqrt{5}+2\sqrt{6}+4\sqrt{30} \\
24 \\
\hline
5 \qquad\qquad +4\sqrt{6}.
\end{array}$$

THIRD OPERATION.

$$\begin{array}{l}
5+4\sqrt{6} \\
-5+4\sqrt{6} \\
\hline
-25-20\sqrt{6} \\
96+20\sqrt{6} \\
\hline
71.
\end{array}$$

(122.) To reduce fractions, having polynomial surds for a numerator or denominator or both, so that either the numerator or denominator may be free from radicals.

Suppose we wish to transform the fraction

$$\frac{1}{\sqrt{3}+\sqrt{2}+1},$$

into an equivalent fraction, having a rational denominator.

It is evident that this transformation can be effected, provided we multiply both numerator and denominator by such a quantity as will cause the denominator to become free of radicals, so that the operation is reduced to the finding a multiplier which will make $\sqrt{3}+\sqrt{2}+1$ rational.

We will first multiply by $-\sqrt{3}+\sqrt{2}+1$.

OPERATION.

$$\begin{array}{rrrrr}
 & \sqrt{3} & +\sqrt{2} & +1 & \\
 & -\sqrt{3} & +\sqrt{2} & +1 & \\
\hline
-3 & -\sqrt{6} & -\sqrt{3} & +\sqrt{2} & \\
+2 & +\sqrt{6} & +\sqrt{3} & +\sqrt{2} & \\
+1 & & & & \\
\hline
 & & & 2\sqrt{2}. &
\end{array}$$

Hence, if we multiply both numerator and denominator of $\frac{1}{\sqrt{3}+\sqrt{2}+1}$ by $-\sqrt{3}+\sqrt{2}+1$ it will become $\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$. Again, multiplying both numerator and denominator of $\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$ by $\sqrt{2}$, we finally have $\frac{\sqrt{2}-\sqrt{6}+2}{4}$. The denominator is now rational.

(123.) Hence, to transform a fraction, having surds in its numerator or denominator or both, into an equivalent fraction, in which the numerator or denominator may be free of surds, we have this

RULE.

Multiply the numerator and denominator by such a quantity as will cause the numerator or denominator, as the required case may be, to become rational.

EXAMPLES.

1. Reduce $\frac{5+\sqrt{3}}{4}$ to a fraction having a rational numerator.

Multiplying both numerator and denominator by $5-\sqrt{3}$, we have

$$\frac{5+\sqrt{3}}{4}=\frac{(5+\sqrt{3})(5-\sqrt{3})}{4(5-\sqrt{3})}=\frac{22}{20-4\sqrt{3}}=\frac{11}{10-2\sqrt{3}}.$$

2. Reduce $\frac{\sqrt{5}+2}{\sqrt{5}+\sqrt{3}+\sqrt{2}}$ to a fraction having a rational denominator.

Multiplying both numerator and denominator by $\sqrt{5}+\sqrt{3}-\sqrt{2}$, we get

$$\frac{5+\sqrt{15}-\sqrt{10}+2\sqrt{5}+2\sqrt{3}-2\sqrt{2}}{6+2\sqrt{15}}.$$

Again, multiplying both numerator and denominator of this last fraction, by $6-2\sqrt{15}$, it becomes

$$\frac{4\sqrt{30}-4\sqrt{15}-6\sqrt{10}+10\sqrt{6}-8\sqrt{3}-12\sqrt{2}}{-24},$$

or changing the signs of both numerator and denominator, it becomes, after striking out the factor 2 from each,

$$\frac{6\sqrt{2}+4\sqrt{3}-5\sqrt{6}+3\sqrt{10}+2\sqrt{15}-2\sqrt{30}}{12}.$$

3. Reduce $\frac{\sqrt{7}-\sqrt{5}}{1+\sqrt{2}}$ to an equivalent fraction having a rational denominator.

Ans. $\frac{\sqrt{14}-\sqrt{10}-\sqrt{7}+\sqrt{5}}{1}$.

4. Reduce $\frac{1}{\sqrt{3}-\sqrt{2}+1}$ to an equivalent fraction having a rational denominator.

Ans. $\frac{2-\sqrt{2}+\sqrt{6}}{4}$.

5. Reduce $\frac{\sqrt{a}+\sqrt{x}}{\sqrt{b}+\sqrt{x}}$, first to a fraction having a rational denominator, and then to a fraction having a rational numerator.

$$\text{Ans.}\begin{cases}\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{b}+\sqrt{x}}=\dfrac{\sqrt{ab}-\sqrt{ax}+\sqrt{bx}-x}{b-x}.\\[2ex]\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{b}-\sqrt{x}}=\dfrac{a-x}{\sqrt{ab}+\sqrt{ax}+\sqrt{bx}-x}.\end{cases}$$

IMAGINARY QUANTITIES.

(124.) We have already shown, that (see Note to the Rule under Art. 96,) an even root of a negative quantity is impossible. Such expressions are called imaginary.

$$\left.\begin{array}{l}\sqrt{-a},\\ \sqrt[4]{-a},\\ \sqrt[6]{-a},\\ \sqrt[2m]{-a},\end{array}\right\}\text{ are all imaginary quantities.}$$

Surd quantities, though their values can not be accurately found, can, nevertheless be approximately obtained; but imaginary quantities can not have their values expressed by any means, either accurately or approximately. They must, therefore, be regarded merely as symbolical expressions.

(125.) We will confine ourselves to the imaginary expressions arising from taking the square root of a negative quantity.

The general form of imaginaries of this kind, is

$$\sqrt{-a}=\sqrt{a\times-1}=\sqrt{a}\times\sqrt{-1},$$

substituting b for $\sqrt{a}$, we have

$$\sqrt{-a}=b\sqrt{-1},$$

so that all imaginary quantities arising from extracting the square root of a minus quantity are of the form

$$b\sqrt{-1}.$$

(126.) If we put $\sqrt{-1} = c$, we shall always have

$$c^2 = -1,$$
$$c^3 = -\sqrt{-1},$$
$$c^4 = 1,$$
$$c^5 = \sqrt{-1}.$$

And in general,

$$c^{4m} = 1,$$
$$c^{4m+1} = \sqrt{-1},$$
$$c^{4m+2} = -1,$$
$$c^{4m+3} = -\sqrt{-1},$$

m being any positive integer whatever.

(127.) From which we easily deduce the following principles.

1. $(+\sqrt{-a})\times(+\sqrt{-a}) = -\sqrt{a^2} = -a.$
2. $(-\sqrt{-a})\times(-\sqrt{-a}) = -\sqrt{a^2} = -a.$
3. $(+\sqrt{-a})\times(-\sqrt{-a}) = +\sqrt{a^2} = +a.$
4. $(+\sqrt{-a})\times(+\sqrt{-b}) = -\sqrt{ab}.$
5. $(-\sqrt{-a})\times(-\sqrt{-b}) = -\sqrt{ab}.$
6. $(+\sqrt{-a})\times(-\sqrt{-b}) = +\sqrt{ab}.$

The above is in accordance with the usual rules for the multiplication of algebraic quantities, and must be considered as a *definition* of this symbol, and of the method of using it, and not as a *demonstration* of its properties.

(128.) The student must not infer from what has been said, that imaginary quantities are useless. So far from being useless, they have lent their aid in the solution of questions, which required the most refined and delicate analysis.

(129.) We will now, in order to become more familiar with the operations of imaginaries, perform some examples in

MULTIPLICATION OF IMAGINARIES.

1. Multiply $4\sqrt{-1}+\sqrt{-2}$ by $2\sqrt{-1}-\sqrt{-3}$.

OPERATION.

$$\begin{array}{l} 4\sqrt{-1}+\sqrt{-2} \\ 2\sqrt{-1}-\sqrt{-3} \\ \hline -8-2\sqrt{2}+4\sqrt{3}+\sqrt{6}. \end{array}$$

2. Multiply $4+\sqrt{-3}$ by $2-\sqrt{-2}$.

OPERATION.

$$\begin{array}{l} 4+\ \sqrt{-3} \\ 2-\ \sqrt{-2} \\ \hline 8+2\sqrt{-3}-4\sqrt{-2}+\sqrt{6}. \end{array}$$

3. Multiply $3-\sqrt{-1}$ by $4+\sqrt{-1}$.

OPERATION.

$$\begin{array}{l} 3-\ \sqrt{-1} \\ 4+\ \sqrt{-1} \\ \hline 12-4\sqrt{-1} \\ \quad +3\sqrt{-1}+1 \\ \hline 12-\ \sqrt{-1}+1=13-\sqrt{-1}. \end{array}$$

4. Multiply $\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ into itself.

OPERATION.

$$\begin{array}{l} \frac{1}{2}-\frac{1}{2}\sqrt{-3} \\ \frac{1}{2}-\frac{1}{2}\sqrt{-3} \\ \hline \frac{1}{4}-\frac{1}{4}\sqrt{-3} \\ \quad -\frac{1}{4}\sqrt{-3}-\frac{3}{4}. \\ \hline \frac{1}{4}-\frac{1}{2}\sqrt{-3}-\frac{3}{4}=-\frac{1}{2}-\frac{1}{2}\sqrt{-3}. \end{array}$$

(130.) We will now perform some examples in

DIVISION OF IMAGINARY QUANTITIES.

1. Divide $4+\sqrt{-2}$ by $2-\sqrt{-2}$.

OPERATION.

$$\frac{4+\sqrt{-2}}{2-\sqrt{-2}}.$$

Multiplying numerator and denominator by $2+\sqrt{-2}$, it becomes

$$\frac{6+6\sqrt{-2}}{6}=1+\sqrt{-2}.$$

In the same way we find

2. $$\frac{1+\sqrt{-1}}{1-\sqrt{-1}}=\sqrt{-1}.$$

3. $$\frac{6\sqrt{-3}}{2\sqrt{-4}}=\tfrac{3}{2}\sqrt{3}.$$

4. $$\frac{3-\sqrt{-1}}{4+2\sqrt{-1}}=\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-1}.$$

(131.) We will also add a couple of examples of the extraction of the square root of imaginary binomial surds.

1. Extract the square root of $3+2\sqrt{-1}$.

Comparing this expression with the general formula (A) Art. 117, we have $a=3$; $b=-4$: hence,

$$\left\{\frac{a+\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{3+\sqrt{13}}{2}\right\}^{\frac{1}{2}}.$$

$$\left\{\frac{a-\sqrt{a^2-b}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{3-\sqrt{13}}{2}\right\}^{\frac{1}{2}}=\left\{\frac{\sqrt{13}-3}{2}\right\}^{\frac{1}{2}}\times\sqrt{-1}.$$

Therefore,

$$\sqrt{3+2\sqrt{-1}}=\left\{\frac{\sqrt{13}+3}{2}\right\}^{\frac{1}{2}}+\left\{\frac{\sqrt{13}-3}{2}\right\}^{\frac{1}{2}}\cdot\sqrt{-1}.$$

2. Extract the square root of $3-2\sqrt{-1}$.

All the difference between this example and the last, is in the sign which connects the two terms, so that we need only change the sign which connects the two terms of the answer to the last, in order to obtain the answer of this. (Compare formulas (A) and (B), Art. 117.)

Hence,

$$\sqrt{3-2\sqrt{-1}}=\left\{\frac{\sqrt{13}+3}{2}\right\}^{\frac{1}{2}}-\left\{\frac{\sqrt{13}-3}{2}\right\}^{\frac{1}{2}}\cdot\sqrt{-1}.$$

If we add the answers of these two questions, we shall have

$$\sqrt{3+2\sqrt{-1}}+\sqrt{3-2\sqrt{-1}}=2\left\{\frac{\sqrt{13}+3}{2}\right\}^{\frac{1}{2}}$$
$$=\sqrt{2(\sqrt{13}+3)}.$$

In the same way we find

$$\sqrt{3+\sqrt{-7}}=\sqrt{\tfrac{7}{2}}+\sqrt{-\tfrac{1}{2}},$$
$$\sqrt{4+3\sqrt{-1}}=3\sqrt{\tfrac{1}{2}}+\sqrt{-\tfrac{1}{2}}.$$

(132.) *Before closing this chapter, we will show the interpretation of the following symbols* $\frac{0}{A}$, $\frac{A}{0}$, $\frac{0}{0}$.

We know from the nature of multiplication, that 0 multiplied by a finite quantity, that is, 0 repeated a finite number of times, must still remain equal 0, hence we have this condition

$$0\times A=0. \qquad (1)$$

Dividing both members of (1) by A, we find

$$0=\frac{0}{A}. \qquad (2)$$

Therefore the symbol $\frac{0}{A}$ will always be equal to 0, as long as A is a finite quantity.

(133.) Since the quotient arising from dividing one number by another becomes greater in proportion as the divisor is diminished, it follows that when the divisor becomes less than any assignable quantity, then the quotient will exceed any assignable quantity. Hence, it is usual for mathematicians to say, that $\frac{A}{0}$ is the representation of an infinite quantity. The symbol employed to represent *infinity* is ∞, so that we have

$$\frac{A}{0}=\infty. \tag{3}$$

(134.) Dividing both members of (1) by 0, we find

$$A=\frac{0}{0}. \tag{4}$$

This being true for all values of A shows that $\frac{0}{0}$ is the symbol of an *indeterminate quantity.*

To illustrate this last symbol, we will take several examples.

1. What is the value of the fraction $\frac{x^2-a^2}{bx-ab}$, when $x=a$?

Substituting a for x, our fraction will become

$$\frac{x^2-a^2}{bx-ab}=\frac{a^2-a^2}{ab-ab}=\frac{0}{0}=\textit{an indeterminate quantity.}$$

If, before substituting a for x, we divide both numerator and denominator of the given fraction by $x-a$, (Art. 55,) we find

$$\frac{x^2-a^2}{bx-ab}=\frac{x+a}{b}.$$

Now, substituting a for x, in this reduced form, we find

$$\frac{x+a}{b}=\frac{a+a}{b}=\frac{2a}{b}.$$

Therefore, $\frac{2a}{b}$ is the true value of $\frac{x^2-a^2}{bx-ab}$, when $x=a$.

2. What is the value of $\frac{x^2-ax}{x^2-2ax+a^2}$, when $x=a$?

Writing a for x, we find

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{a^2-a^2}{a^2-2a^2+a^2}=\frac{0}{0}.$$

If we reduce this fraction by dividing both numerator and denominator by $x-a$, we find

$$\frac{x^2-ax}{x^2-2ax+a^2}=\frac{x}{x-a}.$$

Now, writing a for x, in the reduced form, we find

$$\frac{x}{x-a}=\frac{a}{a-a}=\frac{a}{0}=\infty. \quad \text{(Art. 133.)}$$

3. What is the the value of $\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}$, when $x=a$?

When a is substituted for x, we have

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{a^3-3a^3+3a^3-a^3}{ab-ab}=\frac{0}{0}.$$

Reducing by dividing numerator and denominator by $x-a$, we find

$$\frac{x^3-3ax^2+3a^2x-a^3}{bx-ab}=\frac{x^2-2ax+a^2}{b}.$$

Writing a for x, we have

$$\frac{x^2-2ax+a^2}{b}=\frac{a^2-2a^2+a^2}{b}=\frac{0}{b}=0. \quad \text{(Art. 132.)}$$

(135.) From the above, we conclude that whenever an algebraic fraction is reduced to the form $\frac{0}{0}$, there exists a factor common to both numerator and denominator, which factor becomes zero for the particular value of the unknown quantity made use of. In the foregoing examples there was very little difficulty in discovering this factor.

It is obvious that examples of this kind may be chosen where it would be more difficult to find this factor.

In the fraction,

$$\frac{\sqrt{\frac{1}{2}(a^2+x^2)}-x}{a-x},$$

if $x=a$, we shall have for its value $\frac{0}{0}$ In this case we do not readily discover the factor required; but if we multiply the numerator and denominator each by $\sqrt{\frac{1}{2}(a^2+x^2)}+x$, it will become

$$\frac{\frac{1}{2}(a^2-x^2)}{(a-x)\left(\sqrt{\frac{1}{2}(a^2+x^2)}+x\right)}.$$

We now discover that the factor sought is $a-x$. Dividing numerator and denominator each by $a-x$, it becomes

$$\frac{\frac{1}{2}(a+x)}{\sqrt{\frac{1}{2}(a^2+x^2)}+x}.$$

Now, when $x=a$, this last expression will become $=\frac{1}{2}$.

Hence, we conclude that indeterminate expressions of the above kind, when properly reduced, will take one of the following forms.

$$\frac{A}{B}=\textit{a finite quantity.}$$

$$\frac{0}{A}=0=\textit{no value.}$$

$$\frac{A}{0}=\infty=\textit{an infinite quantity.}$$

CHAPTER V.

QUADRATIC EQUATIONS.

(136.) We have already (Art. 66), defined a *quadratic equation*, to be an equation in which the unknown quantity does not exceed the second degree.

The most general form of a quadratic equation of one unknown quantity, is

$$ax^2 + bx = c. \qquad (1)$$

Dividing all the terms of (1) by a, (Axiome IV,) we find

$$x^2 + \frac{b}{a}x = \frac{c}{a}, \qquad (2)$$

where, if we assume $A = \frac{b}{a}$, and $B = \frac{c}{a}$, we shall have

$$x^2 + Ax = B \qquad (3)$$

Equation (3) is as general a form for quadratics as equation (1).

In (3), A and B can have any values either positive or negative.

(137.) When $A = 0$, equation (3) will become

$$x^2 = B, \qquad (4)$$

which is called an *incomplete quadratic equation*, since one of the terms in the general forms (1) and (3) is wanting.

(138.) When $B=0$, equation (3) will become

$$x^2+Ax=0,$$

which divided by x is reduced to

$$x+A=0,$$

which is no longer a quadratic equation, but a simple equation.

(139.) If $A=0$ and $B=0$ at the same time, equation (3) will become

$$x^2=0,$$

which can only be satisfied by taking $x=0$.

INCOMPLETE QUADRATIC EQUATIONS.

(140.) We have just seen that the general form of an incomplete quadratic equation is

$$x^2=B. \qquad (1)$$

If we extract the square root of both members of this equation, we shall (Art. 96,) have

$$x=\pm\sqrt{B}. \qquad (a)$$

Equation (*a*) may be regarded as a general solution of incomplete quadratic equations.

(141.) To find the value of the unknown, when the equation which involves it, leads to an incomplete quadratic equation, we have this

RULE.

I. *Clear the equation of fractions by the same rule as for simple equations.* (Art. 70.)

II. *Then transpose and unite the like terms, if necessary, observing the rule under Art.* 73, *and we shall thus obtain, after dividing by the coefficient of x^2, an equation of the*

form of $x^2 = B$. Extracting the square root of both members, we shall find $x = \pm\sqrt{B}$.

EXAMPLES.

1. Given $\frac{x^2+2}{19}+7=9$, to find x.

This, when cleared of fractions, by multiplying by 19, becomes

$$x^2+2+133=171,$$

transposing and uniting terms, we find $x^2=36$. If we compare this with our general form, we shall see that $B=36$. Extracting the square root, we have $x=\pm 6$, or as it may be better expressed, $x=6$ or $x=-6$.

2. Given $\frac{3}{14x^2}+\frac{1}{2}=\frac{346}{686}$, to find x.

This cleared of fractions, becomes

$$147+343x^2=346x^2,$$

transposing and uniting terms $\qquad 3x^2=147,$

dividing by 3 $\qquad x^2=\ \ 49,$

extracting the square root, we find $\qquad x=\ \ \pm 7.$

3. Given $x^2-\frac{25x^2}{36}=44$, to find x.

Ans. $x=\pm 12$.

4. Given $8+5x^2=\frac{x^2}{5}+4x^2+28$, to find x.

Ans. $x=\pm 5$.

5. Given $2+\frac{x^2}{3}-7=\frac{x^2}{9}+13$, to find x.

Ans. $x=\pm 9$.

(142.) We must be careful to interpret the double sign $\pm$, correctly, the meaning of which is, that the quantity

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before which it is placed may be either plus, or it may be minus. It does not mean that the quantity can be both plus and minus at the same time.

(143.) If an equation involving one unknown quantity can be reduced to the form $x^n = N$, the value of x can be found by simply extracting the nth root of both members, thus,

$$x = \sqrt[n]{N}.$$

(144.) Where it must be observed (Art. 96.) that when n is an *even* number, the value of x will be either plus or minus for all positive values of N, but for negative values of N the value of x will be impossible. When n is an *odd* number, the value of x will have the same sign as N has.

(145.) If the equation can be reduced to the form $x^{\frac{1}{m}} = N$, then x can be found by raising both members to the mth power, thus: $x = N^m$.

(146.) Where x will be positive for all values of N, provided m is an *even* number, but when m is an *odd* number then x will have the same sign as N.

(147.) Finally, when the equation can be reduced to the form

$$x^{\frac{n}{m}} = N.$$

We must first involve both members to the mth power, and then extract the nth root, or else we may first extract the nth root, and then involve to the mth power. (Art. 98.) Thus,

$$x = N^{\frac{m}{n}}.$$

EXAMPLES.

1. Given $\dfrac{\sqrt{x} + 28}{\sqrt{x} + 4} = \dfrac{\sqrt{x} + 38}{\sqrt{x} + 6}$, to find x.

This, when cleared of fractions, becomes

$$x+34\sqrt{x}+168=x+42\sqrt{x}+152,$$

transposing and uniting terms, we have

$$8\sqrt{x}=16,$$

dividing by 8, $\sqrt{x}=2,$

raising to the second power, $x=4.$

2. Given $\sqrt{x}+\sqrt{a+x}=\dfrac{2a}{\sqrt{a+x}}$, to find x.

This equation, when cleared of the fractions, by multiplying by $\sqrt{a+x}$, becomes

$$\sqrt{ax+x^2}+a+x=2a,$$

$$\sqrt{ax+x^2}=a-x,$$

squaring both members,

$$ax+x^2=a^2-2ax+x^2,$$

$$3ax=a^2$$

$$x=\frac{a}{3}.$$

3. Given $3+x^{\frac{2}{3}}=7$, to find x.

Ans. $x=\pm 8$.

4. Given $(y^n-b)^{\frac{1}{2}}=a-d$, to find y.

Ans. $y=\left\{(a-d)^2+b\right\}^{\frac{1}{n}}$.

5. Given $\sqrt{x-32}=16-\sqrt{x}$, to find x.

Ans. $x=81$.

6. Given $(x+a)^{\frac{1}{2}}=\dfrac{a+b}{(x-a)^{\frac{1}{2}}}$, to find x.

Ans. $x=\pm(2a^2+2ab+b^2)^{\frac{1}{2}}$.

7. Given $\dfrac{\sqrt{x}+\sqrt{x-a}}{\sqrt{x}-\sqrt{x-a}}=\dfrac{ac^2}{x-a}$, to find x.

Ans. $x=\dfrac{a(1\pm c)^2}{1\pm 2c}$.

8. Given $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=\frac{3}{2}\sqrt{\frac{x}{x+\sqrt{x}}}$, to find x.

Ans. $x=\frac{25}{16}$.

9. Given $\frac{1}{1-\sqrt{1-x^2}}-\frac{1}{1+\sqrt{1-x^2}}=\frac{\sqrt{3}}{x^2}$, to find x.

Ans. $x=\pm\frac{1}{2}$.

COMPLETE QUADRATIC EQUATIONS.

(148.) We have already seen, that

$$ax^2+bx=c, \qquad (A)$$

is the most general form of a quadratic equation, where

$a=$ the coefficient of the first term;

$b=$ the coefficient of the second term;

$c=$ the term independent of x.

If we multiply the general quadratic equation (A), by $4a$, it will become

$$4a^2x^2+4abx=4ac. \qquad (1)$$

Adding b^2 to both members of (1), it becomes

$$4a^2x^2+4abx+b^2=b^2+4ac. \qquad (2)$$

The left-hand member of this equation is a complete square, equal to $(2ax+b)^2$. The process by which we so transform an equation as to cause one of its members to become a complete square, is called *Completing the Square*. This may be effected by the following

RULE.

Let the quadratic equation be reduced to this form, $ax^2+bx=c$. Then multiply each member by four times the coefficient of the first term, after which add to each member the square of the coefficient of the second term.

EXAMPLES.

1. Complete the square of the equation $x^2 + 3x = 4$.

Multiplying each member by 4, we have

$$4x^2 + 12x = 16.$$

Adding the square of $3 = 9$, to each member, we find

$$4x^2 + 12x + 9 = 25.$$

The left hand member is now a complete square, equal to $(2x + 3)^2$, so also is the right hand member.

2. Complete the square of $18x^2 - 3x = 1$.

Multiplying each member by $4 \times 18 = 72$, we have

$$1296x^2 - 216x = 72.$$

Adding $3^2 = 9$, to each member we finally have

$$1296x^2 - 216x + 9 = 81,$$

each member of which is a complete square.

3. Complete the square of $6x^2 - 7x = -2$.

Ans. $144x^2 - 168x + 49 = 1$.

4. Complete the square of $10x^2 - 99x = 10$.

Ans. $400x^2 - 3960x + 9801 = 10201$.

Having completed the square of a quadratic equation, if we extract the square root of each member, the result will be a simple equation, but as the square root of a quantity may be either positive or negative, it follows that our result will be equivalent to two distinct simple equations. Thus, returning to our general equation, $ax^2 + bx = c$, which, when its square was completed, became $4a^2x^2 + 4abx + b^2 = b^2 + 4ac$, we have, by extracting the square root of each member,

$$2ax + b = \pm\sqrt{b^2 + 4ac}.$$

If we make use of the + sign, we have

$$2ax + b = \sqrt{b^2 + 4ac}.$$

If we use the — sign, we have

$$2ax + b = -\sqrt{b^2 + 4ac}.$$

Hence, a quadratic equation must, in general, yield two distinct values for the unknown quantity. The above results give at once

$$x = \frac{-b + \sqrt{b^2 + 4ac}}{2a},$$

$$\text{or,}\quad x = \frac{-b - \sqrt{b^2 + 4ac}}{2a}.$$

Uniting these values by the aid of the ambiguous sign $\pm$, which is read *plus* or *minus*, not *plus* and *minus*, we have

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}. \qquad \text{(B)}$$

(149.) This may be regarded as a general solution of all quadratic equations, and it is obvious that we may derive from it a general rule which will apply to all quadratic equations, so as not to be under the necessity of actually going through with all the preliminary steps of *completing the square*. The following is such a

RULE.

Having reduced the equation to the general form $ax^2 + bx = c$, we can find x, by taking the coefficient of the second term with its sign changed, plus or minus the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of x, and the whole divided by twice the coefficient of the first term.

EXAMPLES.

1. Given $4x - \frac{36 - x}{x} = 46$, to find the values of x.

This, when cleared of fractions, becomes

$$4x^2 - 36 + x = 46x.$$

Transposing and uniting terms, we have

$$4x^2 - 45x = 36.$$

This compared with the general form

$$ax^2 + bx = c.$$

gives $a = 4$; $b = -45$; $c = 36$.

The square of the coefficient of the second term

$$= (-45)^2 = 2025.$$

Four times the coefficient of the first term into the term independent of x,

$$= 4 \times 4 \times 36 = 576.$$

Therefore, taking the square root of the square of the coefficient of the second term increased by four times the coefficient of the first term into the term independent of x, we get

$$\pm\sqrt{2025 + 576} = \pm\sqrt{2601} = \pm 51.$$

This added to the coefficient of the second term with the sign changed, gives

$$45 \pm 51,$$

which must be divided by twice the coefficient of the first term. Hence,

$$x = \frac{45 \pm 51}{8}.$$

If we take the upper sign, we get

$$x = \frac{45 + 51}{8} = 12.$$

If we take the lower sign, we find

$$x = \frac{45 - 51}{8} = -\frac{3}{4}.$$

Therefore, $x = 12$, or $-\frac{3}{4}$.

Either of which values of x, will verify the equation.

2. Given $\frac{3x-4}{x-4}=9-\frac{x-2}{2}$, to find the values of x.

This, when reduced to the general form, becomes

$$x^2-18x=-72.$$

Squaring 18, we get

$$(18)^2=324.$$

Four times the first coefficient multiplied into -72, gives

$$4\times-72=-288,$$

which added to 324, gives 36, the square root of which is ± 6.

Therefore, $$x=\frac{18\pm 6}{2}=12 \text{ or } 6.$$

3. Given $\sqrt{3x-5}=\frac{\sqrt{7x^2+36x}}{x}$, to find the values of x.

Squaring both members, we have

$$3x-5=\frac{7x^2+36x}{x^2}=\frac{7x+36}{x}.$$

This, cleared of fractions, becomes

$$3x^2-5x=7x+36.$$

Transposing and uniting terms, we have

$$3x^2-12x=36.$$

This divided by 3, gives

$$x^2-4x=12.$$

Therefore, $x=\frac{4\pm\sqrt{(4)^2+4\times 12}}{2}=\frac{4\pm 8}{2}=6, \text{ or } -2.$

4. Given $\frac{3}{x^2-3x}+\frac{3}{x^2+4x}=\frac{27}{8x}$, to find the values of x.

This, by reduction, becomes

$$9x^2-7x=116.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times9\times116}}{18}=\frac{7\pm65}{18}=4$, or $-3\frac{2}{9}$.

5. Given $\frac{x^2+12}{2}+\frac{x}{2}=4x$, to find x.

This reduced, becomes

$$x^2-7x=-12.$$

Therefore, $x=\frac{7\pm\sqrt{7^2+4\times-12}}{2}=\frac{7\pm1}{2}=4$, or 3.

(150.) An equation of the form

$$ax^{2n}+bx^n=c, \qquad \text{(A)}$$

can be solved by the above rule, which indeed will agree with the form under consideration in the particular case of $n=1$.

If, in the above equation, we write y for x^n, and consequently y^2 for x^{2n}, it will become

$$ay^2+by=c,$$

which is precisely of the form of (A), Art. 148. Consequently,

$$y=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

Re-substituting x^n for y, we have

$$x^n=\frac{-b\pm\sqrt{b^2+4ac}}{2a}.$$

And,
$$x=\left\{\frac{-b\pm\sqrt{b^2+4ac}}{2a}\right\}^{\frac{1}{n}}. \qquad \text{(B)}$$

This value of x, must hold for all values of the constants n, a, b, and c, whether positive or negative, integral or fractional.

EXAMPLES.

1. Given $x^4+ax^2=b$, to find x.

This becomes $y^2+ay=b$, when for x^2 we write y.

$$\therefore y=\frac{-a\pm\sqrt{a^2+4b}}{2}=x^2,$$

hence, $$x=\left\{\frac{-a\pm\sqrt{a^2+4b}}{2}\right\}^{\frac{1}{2}}.$$

2. Given $3x^{2n}-2x^n=8$, to find x.

$$x^n=\frac{2\pm 10}{6}=2,$$

$$\therefore x=\sqrt[n]{2}.$$

3. Given $2(1+x-x^2)-\sqrt{1+x-x^2}=-\frac{1}{9}$, to find x.

If for $1+x-x^2$, we put y^2, our equation will become

$$2y^2-y=-\frac{1}{9},$$

or $$18y^2-9y=-1,$$

$$\therefore y=\frac{9\pm 3}{36}=\frac{1}{3}, \text{ or } \frac{1}{6},$$

hence $$y^2=\frac{1}{9}, \text{ or } \frac{1}{36}.$$

Re-substituting $1+x-x^2$, for y^2, we have, when we take the first value of y^2,

$$1+x-x^2=\frac{1}{9},$$

$$9x^2-9x=8,$$

$$\therefore x=\frac{9\pm 3\sqrt{41}}{18}=\frac{1}{2}+\frac{1}{6}\sqrt{41}, \text{ or } \frac{1}{2}-\frac{1}{6}\sqrt{41}.$$

When we take the other value of y^2, we have

$$1+x-x^2=\frac{1}{36},$$

or $$36x^2-36x=35,$$

$$\therefore x=\frac{36\pm24\sqrt{11}}{72}=\frac{1}{2}+\frac{1}{3}\sqrt{11},\text{ or }\frac{1}{2}-\frac{1}{3}\sqrt{11}.$$

Collecting these four values of x, we find

$$x=\tfrac{1}{2}+\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}-\tfrac{1}{6}\sqrt{41},$$
$$x=\tfrac{1}{2}+\tfrac{1}{3}\sqrt{11},$$
$$x=\tfrac{1}{2}-\tfrac{1}{3}\sqrt{11}.$$

4. Given $\left(x^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}+\left(a^2-\frac{a^4}{x^2}\right)^{\frac{1}{2}}=\frac{x^2}{a}$, to find the values of x.

This equation is easily put under this form

$$\frac{a}{x}\sqrt{a^2x^2-a^4}=x^2-\frac{a}{x}\sqrt{x^4-a^4}.$$

This squared, becomes

$$a^4-\frac{a^6}{x^2}=x^4-2ax\sqrt{x^4-a^4}+a^2x^2-\frac{a^6}{x^2}.$$

By transposing, we have

$$x^4-a^4-2ax\sqrt{x^4-a^4}+a^2x^2=0.$$

Extracting the square root, we find

$$\sqrt{x^4-a^4}-ax=0,$$

or $$\sqrt{x^4-a^4}=ax.$$

Squaring, we find

$$x^4-a^4=a^2x^2,$$

or $$x^4-a^2x^2=a^4.$$

Hence, $$x^2=\frac{a^2\pm a^2\sqrt{5}}{2}.$$

Consequently,

$$x = \pm a\left(\frac{1 \pm \sqrt{5}}{2}\right)^{\frac{1}{2}}.$$

(151.) We have seen that the general form of a quadratic equation, $ax^2 + bx = c$, gave, for the value of the unknown, the following expression:

$$x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

When $a = 1$, the equation $ax^2 + bx = c$, becomes

$$x^2 + bx = c. \qquad \text{(C)}$$

And the above expression for the unknown, will become

$$x = \frac{-b \pm \sqrt{b^2 + 4c}}{2} = -\frac{b}{2} \pm \sqrt{\left(\frac{b}{2}\right)^2 + c}. \qquad \text{(D)}$$

Now, since all quadratic equations may be made to assume the form of (C), by dividing all the terms by the coefficient of x^2, it follows that formula (D) must, when properly translated into common language, give a general rule for the solution of all quadratic equations. The following is the

RULE.

Having reduced the equation to the form $x^2 + bx = c$, we can find x by taking half the coefficient of the second term, with its sign changed; plus or minus the square root of the square of the half of the coefficient of the second term increased by the term independent of x.

EXAMPLES.

1. Given $x^2 - 10x = -24$, to find x.

In this example, half the coefficient of the second term is 5, which squared and added to -24, the term independent of x, is 1. Extracting the square root of 1, we have ± 1.

Therefore, $\qquad x = 5 \pm 1 = 6$, or 4.

2. Given $\frac{x}{x+60}=\frac{7}{3x-5}$, to find x.

This cleared of fractions, becomes

$$3x^2-5x=7x+420.$$

Transposing and uniting terms, we have

$$3x^2-12x=420.$$

Dividing by 3, we have

$$x^2-4x=140,$$

$$\therefore x=2\pm 12=14, \text{ or } -10.$$

3. Given $\frac{x+12}{x}+\frac{x}{x+12}=\frac{26}{5}$, to find x.

Ans. $x=3$, or -15.

4. Given $3x^6+42x^3=3321$, to find x.

Ans. 3, or $(-41)^{\frac{1}{3}}$.

(152.) EQUATIONS CONTAINING TWO OR MORE UNKNOWN QUANTITIES, WHICH INVOLVE IN THEIR SOLUTION QUADRATIC EQUATIONS.

1. Given $\left\{\begin{array}{l} xy=125x+300y \\ y^2-x^2=90000. \end{array}\right\}$, to find x and y.

From the first of these equations, we find

$$x=\frac{300y}{y-125}.$$

Substituting this value of x in the second equation, it becomes

$$y^2-\left\{\frac{300y}{y-125}\right\}^2=90000.$$

Which, when expanded, is

$$y^2-\frac{90000y^2}{y^2-250y+15625}=90000.$$

This, cleared of fractions, and terms united, becomes

$$y^4-250y^3-164375y^2+22500000y=1406250000.$$

This may be written as follows

$$(y^2-125y)^2-180000(y^2-125y)=1406250000.$$

Solving by rule for quadratics, considering y^2-125y as the unknown quantity, we have

$$y^2-125y=90000\pm97500.$$

Hence,

$$y^2-125y=187500, \text{ or } y^2-125y=-7500.$$

The first of these gives

$$y=\frac{125\pm875}{2}=500, \text{ or } -375.$$

The second gives

$$y=\frac{125\pm25\sqrt{-23}}{2}.$$

Both of which values are imaginary.

Having found y, we can substitute it in the equation

$$x=\frac{300y}{y-125},$$

and thus obtain the values of x.

2. Given $\left\{\begin{array}{ll} y^6-x^4=a^2 & (1) \\ x^2y^3-cx^2=ay^3 & (2) \end{array}\right\}$, to find x and y.

From (2), we get

$$x^2=\frac{ay^3}{y^3-c}, \tag{3}$$

which substituted in (1), we have

$$y^6-\frac{a^2y^6}{y^6-2cy^3+c^2}=a^2. \tag{4}$$

Clearing (4) of fractions, it may then be put under the form

$$(y^6-cy^3)^2-2a^2(y^6-cy^3)=a^2c^2. \qquad (5)$$

Solving this by quadratics, considering y^6-cy^3 as the unknown quantity, we have

$$y^6-cy^3=a^2\pm a\sqrt{a^2+c^2}. \qquad (6)$$

Again, solving (6) by quadratics, considering y^3 as the unknown, we have

$$y^3=\frac{c}{2}\pm\left(a^2\pm a\sqrt{a^2+c^2}+\frac{c^2}{4}\right)^{\frac{1}{2}}. \qquad (7)$$

Extracting the cube root of (7), it becomes

$$y=\left\{\frac{c}{2}\pm\left(a^2\pm a\sqrt{a^2+c^2}+\frac{c^2}{4}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}. \qquad (8)$$

The value of y, (8), or better the value of y^3, (7), when substituted in (3), will give x.

3. Given $\left\{\begin{array}{lr} v+w+x+y+z= & 56 \ (1) \\ vw-x-y-z= & 207 \ (2) \\ wx-v-y-z= & -9 \ (3) \\ xy-v-w-z= & -19 \ (4) \\ yz-v-w-x= & 38 \ (5) \end{array}\right\}$, to find v, w, x, y, and z.

$$\begin{array}{ll} vw+v+w=263, & (6)=(1)+(2) \\ wx+w+x=\ 47, & (7)=(1)+(3) \\ xy+x+y=\ 37, & (8)=(1)+(4) \\ yz+y+z=\ 94. & (9)=(1)+(5) \end{array}$$

By adding a unit to both members of equations (6), (7), (8), (9), they may be put under the following forms:

$$\begin{array}{ll} (v+1)(w+1)=264, & (10) \\ (w+1)(x+1)=\ 48, & (11) \\ (x+1)(y+1)=\ 38, & (12) \\ (y+1)(z+1)=\ 95. & (13) \end{array}$$

If we add 5 to both members of (1) it may be written as follows:

$$(v+1)+(w+1)+(x+1)+(y+1)+(z+1)=61. \qquad (14)$$

We shall now use equations (10), (11), (12), (13) and (14), which are symmetrical instead of the original equations.

$$w+1=\frac{264}{v+1}, \qquad (15)=(10)\div(v+1)$$

$$x+1=\frac{48}{w+1}=\frac{2}{11}(v+1), \qquad (16)=(11)\div(w+1)$$

$$y+1=\frac{38}{x+1}=\frac{209}{v+1}, \qquad (17)=(12)\div(x+1)$$

$$z+1=\frac{95}{y+1}=\frac{5}{11}(v+1). \qquad (18)=(13)\div(y+1)$$

Substituting these values of $w+1$, $x+1$, $y+1$, $z+1$, in (14), we have

$$(v+1)+\frac{264}{v+1}+\frac{2}{11}(v+1)+\frac{209}{v+1}+\frac{5}{11}(v+1)=61. \qquad (19)$$

This reduces to this form,

$$\frac{18}{11}(v+1)+\frac{473}{v+1}=61. \qquad (20)$$

Clearing of fractions, we have

$$18(v+1)^2-671(v+1)=-5203. \qquad (21)$$

This quadratic solved, gives

$$v+1=11, \text{ or } 26\tfrac{5}{18}.$$

These two values of $v+1$, being substituted in (15) (16), (17), (18), will give two sets of values for $w+1$ $x+1$, $y+1$, $z+1$. These values when found are,

$$v+1=11, \text{ or } 26\tfrac{5}{18}.$$
$$w+1=24, \quad 10\tfrac{2}{43}.$$

$$x + 1 = 2, \text{ or } 4\tfrac{7}{8}.$$
$$y + 1 = 19, \text{ or } 7\tfrac{41}{43}.$$
$$z + 1 = 5, \text{ or } 11\tfrac{15}{17}.$$

Consequently,

$$\left.\begin{cases} v = 10, \\ w = 23, \\ x = 1, \\ y = 18, \\ z = 4, \end{cases}\right\} \quad \text{or} \quad \left.\begin{cases} v = 25\tfrac{5}{18}. \\ w = 9\tfrac{2}{43}. \\ x = 3\tfrac{7}{8}. \\ y = 6\tfrac{41}{43}. \\ z = 10\tfrac{1}{1}{}^{7}. \end{cases}\right\}$$

4. Given $x^{4n} - 2x^{3n} + x^n = 6$, to find x.

This is readily put under this form

$$(x^{2n} - x^n)^2 - (x^{2n} - x^n) = 6. \qquad (1)$$

If we make $y = x^{2n} - x^n$, equation (1) will become

$$y^2 - y = 6, \qquad (2)$$

$$\therefore y = \tfrac{1}{2} \pm \tfrac{5}{2}. \qquad (3)$$

Re-substituting for y, we have

$$x^{2n} - x^n = 3, \qquad (4)$$

or

$$x^{2n} - x^n = -2. \qquad (5)$$

Now, in (4) and (5) substituting z for x^n, and we have

$$z^2 - z = 3, \qquad (6)$$

or

$$z^2 - z = -2. \qquad (7)$$

From (6), we have

$$z = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{13}. \qquad (8)$$

From (7), we find

$$z = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-7}. \qquad (9)$$

Re-substituting x^n for z, we find

$$x^n = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{13}, \qquad (10)$$

$$x^n = \tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-7}. \qquad (11)$$

Taking the nth roots of (10) and (11), we find

$$\text{Ans.} \begin{cases} x = \left\{\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{13}\right\}^{\frac{1}{n}}, \\ x = \left\{\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{-7}\right\}^{\frac{1}{n}}. \end{cases}$$

5. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^2+y^2=b & (2) \end{array}\right\}$, to find x and y.

Squaring (1), we have

$$x^2+2xy+y^2=a^2. \qquad (3)$$

Subtracting (2) from (3), we get

$$2xy=a^2-b. \qquad (4)$$

Subtracting (4) from (2), we find

$$x^2-2xy+y^2=2b-a^2. \qquad (5)$$

Extracting the square root of (5), we get

$$x-y=\pm\sqrt{2b-a^2}. \qquad (6)$$

Taking half the sum of (1) and (6), we get

$$x=\frac{a}{2}\pm\frac{1}{2}\sqrt{2b-a^2}. \qquad (7)$$

Subtracting (7) from (1), we find

$$y=\frac{a}{2}\mp\frac{1}{2}\sqrt{2b-a^2}. \qquad (8)$$

6. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^3+y^3=b & (2) \end{array}\right\}$, to find x and y.

We will indicate our operations upon the successive equations, by the method explained under Art. 80.

$$x^3+3x^2y+3xy^2+y^3=a^3. \qquad (3)=(1)^3$$

$$3xy(x+y)=a^3-b. \qquad (4)=(3)-(2)$$

$$3xy=\frac{a^3-b}{a}. \qquad (5)=(4)\div(1)$$

$$xy=\frac{a^3-b}{3a}. \qquad (6)=(5)\div 3$$

$$x^2+2xy+y^2=a^2. \qquad (7)=(1)^2$$

$$4xy=\frac{4a^3-4b}{3a}. \qquad (8)=(6)\times 4$$

$$x^2-2xy+y^2=\frac{4b-a^3}{3a}. \qquad (9)=(7)-(8)$$

$$x-y=\left\{\frac{4b-a^3}{3a}\right\}^{\frac{1}{2}}. \qquad (10)=\sqrt{(9)}$$

$$x=\frac{a}{2}\pm\frac{1}{2}\left\{\frac{4b-a^3}{3a}\right\}^{\frac{1}{2}}. \qquad (11)=\frac{(1)+(10)}{2}$$

$$y=\frac{a}{2}\mp\frac{1}{2}\left\{\frac{4b-a^3}{3a}\right\}^{\frac{1}{2}}. \qquad (12)=\frac{(1)-(10)}{2}$$

7. Given $\left\{\begin{array}{ll} x+y=a & (1) \\ x^4+y^4=b & (2) \end{array}\right\}$, to find x and y.

$$x^4+4x^3y+6x^2y^2+4xy^3+y^4=a^4. \qquad (3)=(1)^4$$

$$4xy(x^2+y^2)+6x^2y^2=a^4-b. \qquad (4)=(3)-(2)$$

$$x^2+2xy+y^2=a^2. \qquad (5)=(1)^2$$

Transposing $2xy$ of (5) we get

$$x^2+y^2=a^2-2xy. \qquad (6)$$

Substituting this value of x^2+y^2 in (4), we get

$$4xy(a^2-2xy)+6x^2y^2=a^4-b. \qquad (7)$$

This becomes, by putting z for xy, and transposing,

$$2z^2-4a^2z=b-a^4, \qquad (8)$$

$$\therefore z=a^2\pm\sqrt{\frac{b+a^4}{2}}. \qquad (9)$$

Hence,

$$xy=a^2\pm\sqrt{\frac{b+a^4}{2}}. \qquad (10)$$

$$4xy=4a^2\pm4\sqrt{\frac{b+a^4}{2}}. \qquad (11)=(10)\times4$$

$$x^2-2xy+y^2=-3a^2\mp 4\sqrt{\frac{b+a^4}{2}}. \quad (12)=(5)-(11)$$

$$x-y=\pm\left\{-3a^2\mp 4\sqrt{\frac{b+a^4}{2}}\right\}^{\frac{1}{2}}. \quad (13)=\surd(12)$$

$$x=\frac{a}{2}\pm\frac{1}{2}\left\{-3a^2\mp 4\sqrt{\frac{b+a^4}{2}}\right\}^{\frac{1}{2}}. \quad (14)=\frac{(1)+(13)}{2}$$

$$y=\frac{a}{2}\mp\frac{1}{2}\left\{-3a^2\mp 4\sqrt{\frac{b+a^4}{2}}\right\}^{\frac{1}{2}}. \quad (15)=\frac{(1)-(13)}{2}$$

8. Given $\left\{\begin{array}{ll} x^2-y^2=a & (1) \\ x^3y+xy^3=b & (2) \end{array}\right\}$, to find x, and y.

$$x^2+y^2=\frac{b}{xy}. \qquad (3)=(2)\div xy$$

$$x^2=\frac{b}{2xy}+\frac{a}{2}. \qquad (4)=\frac{(3)+(1)}{2}$$

$$y^2=\frac{b}{2xy}-\frac{a}{2}. \qquad (5)=\frac{(3)-(1)}{2}$$

$$x^2y^2=\frac{b^2}{4x^2y^2}-\frac{a^2}{4}. \qquad (6)=(4)\times(5)$$

This readily gives,

$$4x^4y^4+a^2x^2y^2=b. \qquad (7)$$

Consequently,

$$xy=\pm\left(\frac{-a^2\pm\sqrt{a^4+16b}}{8}\right)^{\frac{1}{2}}. \qquad (8)$$

This value of xy, introduced into (4) and (5), we obtain

$$x=\pm\left\{\pm b\left(\frac{2}{-a^2\pm\sqrt{a^4+16b}}\right)^{\frac{1}{2}}+\frac{a}{2}\right\}^{\frac{1}{2}}. \qquad (9)$$

$$y=\pm\left\{\pm b\left(\frac{2}{-a^2\pm\sqrt{a^4+16b}}\right)^{\frac{1}{2}}-\frac{a}{2}\right\}^{\frac{1}{2}}. \quad (10)$$

9. Given $\left\{\begin{array}{ll} x^5y^3+x^3y^5=135 & (1) \\ x^{10}y^6+x^6y^{10}=5103 & (2) \end{array}\right\}$, to find x and y.

$$\begin{array}{rl}
x^{10}y^6+2x^8y^8+x^6y^{10}=18225 & (3)=(1)^2 \\
2x^8y^8=13122 & (4)=(3)-(2) \\
x^8y^8=6561 & (5)=(4)\div 2 \\
xy=\pm 3 & (6)=\sqrt[8]{(5)} \\
x^3y^3=\pm 27 & (7)=(6)^3 \\
x^2+y^2=\pm 5 & (8)=(1)\div(7) \\
2xy=\pm 6 & (9)=(6)\times 2 \\
x^2-2xy+y^2=\mp 1 & (10)=(8)-(9) \\
x-y=\pm\sqrt{-1} \text{ or } \pm 1 & (11)=\sqrt{(10)} \\
x^2+2xy+y^2=\pm 11 & (12)=(8)+(9) \\
x+y=\pm\sqrt{11} \text{ or } \pm\sqrt{-11} & (13)=\sqrt{(12)}
\end{array}$$

$$x=\tfrac{1}{2}(\pm\sqrt{11}\pm\sqrt{-1}) \text{ or } \tfrac{1}{2}(\pm\sqrt{-11}\pm 1) \quad (14)=\frac{(13)+(11)}{2}$$

$$y=\tfrac{1}{2}(\pm\sqrt{11}\mp\sqrt{-1}) \text{ or } \tfrac{1}{2}(\pm\sqrt{-11}\mp 1) \quad (15)=\frac{(13)-(11)}{2}$$

10. Given $\left\{\begin{array}{ll} yx^3-x^3+x=3 & (1) \\ yx(yx^2+1)-x^3+x=6 & (2) \end{array}\right\}$, to find x and y.

Subtracting (1) from (2), we find

$$y^2x^3+yx-yx^3=3. \quad (3)$$

Dividing (3) by (1), we get

$$y=1. \quad (4)$$

This value of y substituted in (1), gives

$$x=3.$$

11. Given $\left\{\begin{array}{ll} x^2y^4-8x^2y^2+16x^2 & \\ \quad =90xy+60(x-y^2)-720(y-1) & (1) \\ \dfrac{(y^2-4y+4)x}{5}=3-\dfrac{12}{x} & (2) \end{array}\right\}$,

to find x and y.

Multiplying (2) by $5x(y^2+4y+4)$, it becomes

$$\left\{\begin{array}{l} x^2y^4-8x^2y^2+16x^2 \\ \qquad =15xy^2+60xy+60x-60y^2-240y-240 \end{array}\right\} \quad (3)$$

Subtracting (1) from (3), we have

$$0=15xy^2-30xy+480y-960. \qquad (4)$$

Dividing (4) by $15xy+480$, it becomes

$$0=y-2, \qquad (5)$$

$$\therefore y=2. \qquad (6)$$

This value of y substituted in (2), gives

$$x=4. \qquad (7)$$

12. Given $\left\{\begin{array}{ll} xy+z=5 & (1) \\ xyz+z^2=15 & (2) \\ xy^2+x^2y-2x+2z=8 & (3) \end{array}\right\}$, to find x, y and z.

Dividing (2) by (1), we find

$$z=3. \qquad (4)$$

Substituting this value of z in (1) and (3) and they become

$$xy=2. \qquad (5)$$

$$xy(x+y)=2+2x. \qquad (6)$$

Dividing (6) by (5), we find

$$x+y=1+x, \qquad (7)$$

$$\therefore y=1. \qquad (8)$$

Dividing (5) by (8), we get

$$x=2. \qquad (9)$$

13. Given $\left\{\begin{array}{ll} x(y+z)=a & (1) \\ y(x+z)=b & (2) \\ z(x+y)=c & (3) \end{array}\right\}$, to find x, y, and z.

Before proceeding to the solution of these equations, we will remark, that they are symmetrical, and consequently

all the derived equations will either contain all the letters similarly combined, or else they will appear in systems of three equations each, which can be deduced from each other by simply permuting.

If we take the sum of (1), (2), and (3), after expanding them, we shall have

$$2xy + 2yz + 2zx = a + b + c. \tag{4}$$

In this equation all the letters enter symmetrically; therefore it will not give rise to any new equation by permutation.

If we subtract twice (3) from (4), we get

$$2xy = a + b - c. \tag{5}$$

By permutation, we derive from (5) these two equations:

$$2yz = b + c - a. \tag{6}$$

$$2zx = c + a - b. \tag{7}$$

Equations (5), (6), and (7) readily give

$$xy = \frac{a + b - c}{2}. \tag{8}$$

$$yz = \frac{b + c - a}{2}. \tag{9}$$

$$zx = \frac{c + a - b}{2}. \tag{10}$$

Taking the continued product of (8), (9) and (10), we have

$$x^2y^2z^2 = \left\{\frac{a+b-c}{2}\right\} \times \left\{\frac{b+c-a}{2}\right\} \times \left\{\frac{c+a-b}{2}\right\}. \tag{11}$$

This equation containing all the letters symmetrically combined, can give no new condition by permutation.

Dividing (11) by the square of (9), we have

$$x^2 = \frac{(a+b-c)(c+a-b)}{2(b+c-a)}. \tag{12}$$

By permuting, we derive from (12) these two equations:

$$y^2=\frac{(b+c-a)(a+b-c)}{2(c+a-b)}. \tag{13}$$

$$z^2=\frac{(c+a-b)(b+c-a)}{2(a+b-c)}. \tag{14}$$

Taking the square roots of (12), (13), and (14), we find

$$x=\pm\left\{\frac{(a+b-c)(c+a-b)}{2(b+c-a)}\right\}^{\frac{1}{2}}. \tag{15}$$

$$y=\pm\left\{\frac{(b+c-a)(a+b-c)}{2(c+a-b)}\right\}^{\frac{1}{2}}. \tag{16}$$

$$z=\pm\left\{\frac{(c+a-b)(b+c-a)}{2(a+b-c)}\right\}^{\frac{1}{2}}. \tag{17}$$

This question is a good illustration of the beautiful method of deriving one quantity from another, of a similar nature, by simply permutating.

14. Given, the two equations

$$(x'+x'')(1+x'x''+x'^2x''+x'x''^2+x'^2x''^2)+x'x''=a,$$

$$\left.\begin{array}{r} x'x''(x'+x'')(x'+x''+x'x'')(x'+x''+x'x''+x'^2x''+x'x''^2) \\ =b, \end{array}\right\}$$

to find x' and x''.

If, in these equations, we make successively the substitutions $x'+x''=y'$, $x'x''=y''$; $y'+y''=z'$, $y'y''=z''$; $z'+z''=w'$, $z'z''=w''$, we shall finally have

$$w'+w''=a,$$
$$w'w''=b.$$

The quantities sought, x', x'', will be determined by means of these four quadratic equations:

$$w^2-aw+b=0.$$
$$z^2-w'z+w''=0.$$

$$y^2 - z'y + z'' = 0.$$
$$x^2 - y'x + y'' = 0.$$

The first of these equations determines w' and w''; the second z' and z''; the third y' and y''; and, finally, the fourth x' and x''. We thus successively obtain

$$w' = \frac{a \pm \sqrt{a^2 - 4b}}{2}, \qquad w'' = \frac{a \mp \sqrt{a^2 - 4b}}{2};$$

$$z' = \frac{w' \pm \sqrt{w'^2 - 4w''}}{2}, \qquad z'' = \frac{w' \mp \sqrt{w'^2 - 4w''}}{2};$$

$$y' = \frac{z' \pm \sqrt{z'^2 - 4z''}}{2}, \qquad y'' = \frac{z' \mp \sqrt{z'^2 - 4z''}}{2};$$

$$x' = \frac{y' \pm \sqrt{y'^2 - 4y''}}{2}, \qquad x'' = \frac{y' \mp \sqrt{y'^2 - 4y''}}{2};$$

and there are, consequently, for x', as well as for x'', sixteen different values. If we had solved the first two equations by the common method, we should, after a laborious elimination, have obtained an equation of the 16th degree.

If $a = 371$, and $b = 13530$, then will one set of values be, $x' = 2$ and $x'' = 3$.

15. Given $\left\{\begin{array}{ll} x^2 + xy + y^2 = a^2, & (1) \\ y^2 + yz + z^2 = b^2, & (2) \\ z^2 + zx + x^2 = c^2, & (3) \end{array}\right\}$ to find x, y, z.

$$\left.\begin{array}{l} 2(x^2+y^2+z^2)+(xy+yz+zx) \\ \qquad = a^2+b^2+c^2. \end{array}\right\} (4)=(1)+(2)+(3)$$

$$\left.\begin{array}{l} 4(x^2+y^2+z^2)^2+4(x^2+y^2+z^2)(xy+yz+zx) \\ \qquad +(xy+yz+zx)^2=(a^2+b^2+c^2)^2. \end{array}\right\} (5)=(4)^2$$

$$x^4 + 3x^2y^2 + y^4 + 2x^3y + 2y^3x = a^4. \qquad (6)=(1)^2$$
$$y^4 + 3y^2z^2 + z^4 + 2y^3z + 2z^3y = b^4. \qquad (7)=(2)^2$$
$$z^4 + 3z^2x^2 + x^4 + 2z^3x + 2x^3z = c^4. \qquad (8)=(3)^2$$

$$\left.\begin{array}{l} 4(x^2+y^2+z^2)^2+4(x^2+y^2+z^2)(xy+yz+zx) \\ \qquad -2(xy+yz+zx)^2=2(a^4+b^4+c^4). \end{array}\right\} \begin{array}{l} (9)=2(6) \\ +2(7)+2(8) \end{array}$$

$$\left.\begin{array}{l}3(xy+yz+zx)^2 \\ \qquad =(a^2+b^2+c^2)^2-2(a^4+b^4+c^4).\end{array}\right\} \quad (10)=(5)-(9)$$

or, which is the same thing,

$$(xy+yz+zx)^2=\tfrac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\tfrac{1}{3}(a^4+b^4+c^4). \quad (11)$$

$$\left.\begin{array}{l}(xy+yz+zx)= \\ \pm\sqrt{\tfrac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\tfrac{1}{3}(a^4+b^4+c^4)}=k.\end{array}\right\} \quad (12)=\sqrt{(11)}$$

$$6\,(xy+yz+zx)=6k. \qquad (13)=(12)\times 6$$

$$\left.\begin{array}{l}4(x^2+y^2+z^2)+2(xy+yz+zx) \\ \qquad =2(a^2+b^2+c^2).\end{array}\right\} \quad (14)=(4)\times 2$$

$$4(x+y+z)^2=2(a^2+b^2+c^2)+6k. \qquad (15)=(14)+(13)$$

$$2(x+y+z)=\pm\sqrt{2(a^2+b^2+c^2)+6k}. \qquad (16)=\sqrt{(15)}$$

$$\left.\begin{array}{l}2(x^2+y^2+z^2+xy+yz+zx) \\ \qquad =a^2+b^2+c^2+k.\end{array}\right\} \quad (17)=(4)+(12)$$

$$2x(x+y+z)=a^2-b^2+c^2+k. \qquad (18)=(17)-(2)\times 2$$

$$x=\frac{a^2-b^2+c^2+k}{\pm\sqrt{2(a^2+b^2+c^2)+6k}}. \qquad (19)=(18)\div(16)$$

Having found the value of x, we may find the values of y and z, by simply permuting the letters in the above expression, (19). Since the expression for k is symmetrical, it must remain constantly the same. Consequently the denominator of the expression for x, (19), will not, during this permutation, change its value.

In this way we find

$$y=\frac{b^2-c^2+a^2+k}{\pm\sqrt{2(a^2+b^2+c^2)+6k}}. \qquad (20)$$

$$z=\frac{c^2-a^2+b^2+k}{\pm\sqrt{2(a^2+b^2+c^2)+6k}}. \qquad (21)$$

If we restore the value of k, we shall have,

$$x=\frac{a^2-b^2+c^2\pm\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}{\pm\sqrt{2(a^2+b^2+c^2)\pm6\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}}.$$

$$y=\frac{b^2-c^2+a^2\pm\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}{\pm\sqrt{2(a^2+b^2+c^2)\pm6\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}}.$$

$$z=\frac{c^2-a^2+b^2\pm\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}{\pm\sqrt{2(a^2+b^2+c^2)\pm6\sqrt{\frac{2}{3}(a^2b^2+b^2c^2+c^2a^2)-\frac{1}{3}(a^4+b^4+c^4)}}}.$$

These expressions show that there are four sets of values for x, y, and z.

We have chosen this example, partly from its being one rather difficult of solution by the ordinary methods, and partly because it affords an excellent opportunity for exemplifying the beauty of symmetrical equations. Equations (1), (2), and (3), which are given, are not only symmetrical, but they are also homogeneous. Consequently all our derived equations will be homogeneous, and will either contain all the different letters similarly involved, as in (4), (5), (9), (10), (11), (12), (13), (14), (15), (16), (17), and (18), or else there will be a system of three equations which can be deduced from each other simply by permutating the letters, as is the case with the given equations (1), (2), and (3), also equations (6), (7), and (8). Equations (19), (20), and (21), are also of this nature. This perfect symmetry of expressions, must in a great measure serve as a check upon our work, preventing errors which otherwise could not be so readily detected.

(153.) Questions which require for their solution a knowledge of quadratic equations.

1. A widow possessed 13,000 dollars, which she divided into two parts, and placed them at interest, in such a manner that the incomes from them were equal. If she had put out the first portion at the same rate as the second, she would have drawn for this part 360 dollars interest; and if she had placed the second out at the same rate as the first, she would have drawn for it 490 dollars interest. What were the two rates of interest?

Let $x=$ the rate per cent. of the first part.

Let $y=$ the rate per cent. of the second part.

Now, since the incomes from the two parts were equal,

they must have been to each other reciprocally as x to y. Hence, if my denote the first part, then will mx denote the second part.

We shall then have

$$m(x+y)=13000.$$

Consequently, $$m=\frac{13000}{x+y}.$$

Therefore, $$\frac{13000y}{x+y}=\text{the first part.}$$

$$\frac{13000x}{x+y}=\text{the second part.}$$

The interest on these parts, at y and x per cent., respectively, is

$$\frac{130y^2}{x+y} \text{ and } \frac{130x^2}{x+y}.$$

Hence, by the conditions of the question, we have

$$\frac{130y^2}{x+y}=360. \qquad (1)$$

$$\frac{130x^2}{x+y}=490. \qquad (2)$$

Dividing (2) by (1), we get

$$\frac{x^2}{y^2}=\frac{49}{36}. \qquad (3)$$

Extracting the square root of (3), we have

$$\frac{x}{y}=\frac{7}{6}. \qquad (4)$$

Subtracting (1) from (2), we have

$$\frac{130(x^2-y^2)}{x+y}=130. \qquad (5)$$

Dividing both numerator and denominator, of the left-hand

member of (5), by $x+y$, and also dividing both members by 130, we get

$$x-y=1. \qquad (6)$$

Dividing (6) by y, we find

$$\frac{x}{y}-1=\frac{1}{y}. \qquad (7)$$

Subtracting (7) from (4), we have

$$1=\frac{7}{6}-\frac{1}{y}. \qquad (8)$$

Clearing (8) of fractions, we obtain

$$6y=7y-6. \qquad (9)$$

$$\therefore\ y=6. \qquad (10)$$

Adding (10) and (6), we get

$$x=7. \qquad (11)$$

Therefore the per cent. of the first part was 7, and of the second part was 6.

2. A certain capital is out at 4 per cent.; if we multiply the number of dollars in the capital, by the number of dollars in the interest for 5 months, we obtain \$117041$\frac{2}{3}$. What is the capital? Ans. \$2650.

3. There are two numbers, one of which is greater than the other by 8, and whose product is 240. What numbers are they? Ans. 12 and 20.

4. The sum of two numbers is $=a$, their product $=b$. What numbers are they?

Ans. $\dfrac{a+\sqrt{(a^2-4b)}}{2}, \dfrac{a-\sqrt{(a^2-4b)}}{2}$.

5. It is required to find a number such, that if we multiply its third part by its fourth, and to the product add 5

times the number required, the sum exceeds the number 200 by as much as the number sought is less than 280.

Ans. 48.

6. A person being asked his age, answered, "My mother was 20 years old when I was born, and her age multiplied by mine, exceeds our united ages by 2500." What was his age? Ans. 42.

7. Determine the fortunes of three persons, A, B, C, from the following data: For every \$5 which A possesses, B has \$9, and C \$10. Farther, if we multiply A's money (expressed in dollars, and considered merely as a number) by B's, and B's money by C's, and add both products to the united fortunes of all three, we shall get 8832. How much had each?

Ans. A \$40, B \$72, C \$80.

8. A person buys some pieces of cloth, at equal prices, for \$60. Had he got three more pieces for the same sum, each piece would have cost him \$1 less. How many pieces did he buy? Ans. 12.

9. Two travellers, A and B, set out at the same time, from two different places, C and D; A, from C to D; and B, from D to C. On the way they met, and it then appears that A had already gone 30 miles more than B, and, according to the rate at which they travel, A calculates that he can reach the place D in 4 days, and that B can arrive at the place C in 9 days. What is the distance between C and D? Ans. 150 miles.

10. The sum of two numbers is 10, and the sum of their fifth powers 17050. What are the numbers?

Ans. 3 and 7.

11. The sum of two numbers is 47, and their product 546. Required the sum of their squares.

Ans. 1117.

12. The sum of two numbers is 20, and their product 99. Required the sum of their cubes.

Ans. 2060.

13. Divide the number a into two such parts, that the sum of their reciprocals may equal b. What are the parts?

$$\text{Ans.}\begin{cases}\dfrac{a}{2}+\left(\dfrac{a^2}{4}-\dfrac{a}{b}\right)^{\frac{1}{2}}.\\[2ex] \dfrac{a}{2}-\left(\dfrac{a^2}{4}-\dfrac{a}{b}\right)^{\frac{1}{2}}.\end{cases}$$

14. Divide $\dfrac{9}{2}$ into two such parts, that the sum of their reciprocals may equal 1. What are the parts?

Ans. 3 and $\dfrac{3}{2}$.

15. Given the sum of the squares of two numbers $=a$, and the sum of their reciprocals $=b$; to determine the numbers.

$$\text{Ans.}\begin{cases}\text{Sum of numbers}=\dfrac{1}{b}\left[ab^2+2\pm2(ab^2+1)^{\frac{1}{2}}\right]^{\frac{1}{2}}.\\[2ex] \text{Difference} \quad \text{“} \quad =\dfrac{1}{b}\left[ab^2-2\mp2(ab^2+1)^{\frac{1}{2}}\right]^{\frac{1}{2}}.\end{cases}$$

16. Find the values of x from the equation

$$\frac{3x+25}{4}-\frac{7}{21+2x}=7+x.$$

Ans. $x=-7$, or $-6\frac{1}{2}$.

17. Find the values of x from the equation

$$\frac{x^2-4x}{13}+\frac{3x^2+11.}{4}=3x-8.$$

Ans. $x=2\pm\sqrt{-1}$.

18. A and B can together perform a piece of work in two days, and it would take A, alone, three days longer to perform it than it would B alone. In what time can A and B respectively perform it?

Ans. $\begin{cases} \text{A would require 6 days.} \\ \text{B \quad `` \quad `` \quad 3 \quad ``} \end{cases}$

19. A, B, and C agree to contribute \$730 towards building a school-house, which is to be at the distance of 2 miles from A, and $\frac{5}{8}$ of a mile further from C than from B. They agree that their shares shall be reciprocally proportional to their distances from the school-house. When it was found that A paid \$98 more than B paid. What was B's distance from the school-house? Ans. $2\frac{7}{8}$ miles.

20. I have a certain number in my thoughts; this I multiply by $2\frac{1}{3}$, add 7 to the product, multiply this sum by 8 times the number; I then divide by 14, and from the quotient subtract four times the number, and thus obtain 2352. What number is it? Ans. 42.

21. Find two numbers such, that their sum and product together may be $=34$, and the sum of their squares exceed the sum of the numbers themselves by 42. What are the numbers?

Ans. $\begin{cases} 4 \text{ and } 6;\ \text{or,} \\ \frac{1}{2}(-11+\sqrt{-59}),\ \frac{1}{2}(-11-\sqrt{-59}). \end{cases}$

22. It is required to find a number, consisting of three digits, such, that the sum of the squares of the digits, without considering their position, may be $=104$; but the

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square of the middle digit exceeds twice the product of the other two by 4; further, if 594 be subtracted from the number sought, the three digits become inverted. What number is it? Ans. 862.

23. Find two numbers such, that their sum, their product, and the difference of their squares may be equal.

Ans. $\frac{3}{2}\pm\frac{1}{2}\sqrt{5}$, $\frac{1}{2}\pm\frac{1}{2}\sqrt{5}$.

24. What two numbers are they, whose sum is 3, and the sum of whose fourth powers is 17?

$$\text{Ans.}\begin{cases} 2 \text{ and } 1; \text{ or,} \\ \frac{1}{2}(3+\sqrt{-55}), \text{ and } \frac{1}{2}(3-\sqrt{-55}). \end{cases}$$

25. What two numbers are they, whose product is 3, and the sum of whose fourth powers is 82?

$$\text{Ans.}\begin{cases} \pm 1, \text{ and } \pm 3; \text{ or,} \\ \pm\sqrt{-1}, \text{ and } \mp\sqrt{-9}. \end{cases}$$

26. A and B can together perform a piece of work in 3 days, and it would take B alone to do it 8 days longer than it would take A. How many days would A alone require to perform it? Ans. 4 days.

Properties of the roots of quadratic equations.

(154.) We have seen that all quadratic equations can be reduced to this general form.

$$x^2+ax=b. \tag{1}$$

This, when solved, gives

$$x=-\frac{a}{2}\pm\sqrt{\frac{a^2}{4}+b}. \tag{2}$$

Therefore the two values of x are

$$-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}. \qquad (3)$$

$$-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}. \qquad (4)$$

(155.) Now, since $\frac{a^2}{4}=\left(\frac{a}{2}\right)^2$ is always positive for all real values of a, it follows that the sign of the expression $\frac{a^2}{4}+b$, depends upon the value of b.

(156.) When b is positive, or when b is negative and less than $\frac{a^2}{4}$, then will $\frac{a^2}{4}+b$ be positive, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be *real*.

(157.) When b is negative, and numerically greater than $\frac{a^2}{4}$, then $\frac{a^2}{4}+b$ will be negative, and consequently $\sqrt{\frac{a^2}{4}+b}$ will be *imaginary*.

CASE I.

When $\sqrt{\frac{a^2}{4}+b}$ *is real.*

1. If a is positive, and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real* and *negative*

2. When a is either positive or negative, and $\frac{a}{2}$ is numerically less than $\sqrt{\frac{a^2}{4}+b}$, then will both values of x be *real*, the one *positive* and the other *negative*.

3. When a is negative and $\frac{a}{2}$ is numerically greater than $\sqrt{\frac{a^2}{4}+b}$, then both values of x will be *real* and *positive*.

CASE II.

When $\sqrt{\frac{a^2}{4}+b}$ *is imaginary.*

In this case both values of x are *imaginary* for all values of a.

(158.) When b is negative, and numerically equal to $\frac{a^2}{4}$, then both values of x become $=-\frac{a}{2}$.

(159.) If we add together the two values of x, we have

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)+\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-a.$$

If we multiply them, we find

$$\left(-\frac{a}{2}+\sqrt{\frac{a^2}{4}+b}\right)\times\left(-\frac{a}{2}-\sqrt{\frac{a^2}{4}+b}\right)=-b.$$

From which we see,

That the sum of the roots of the quadratic equation $x^2+ax=b$ *is equal to* $-a$.

And the product of the roots is equal to $-b$.

Hence the roots of the equation.

$$x^2-(r_1+r_2)x=-r_1r_2,$$

are r_1 and r_2.

(160.) We can also deduce these properties as follows:

If, in the equation $x^2+ax=b$, we suppose the two roots of x to be r_1 and r_2, we shall have

$$r_1^2 + ar_1 = b. \qquad (1)$$
$$r_2^2 + ar_2 = b. \qquad (2)$$

Subtracting (1) from (2), we find

$$r_2^2 - r_1^2 + a(r_2 - r_1) = 0. \qquad (3)$$

Dividing (3) by $r_2 - r_1$, it becomes

$$r_2 + r_1 + a = 0; \qquad (4)$$
$$\therefore r_2 + r_1 = -a. \qquad (5)$$

Multiplying (4) by r_1, we get

$$r_2 r_1 + r_1^2 + ar_1 = 0. \qquad (6)$$

Subtracting (1) from (6), we get

$$r_2 r_1 = -b. \qquad (7)$$

Equations (5) and (7) correspond with the properties just found, Art. 159.

(161.) We have seen that every quadratic equation, when solved, gives two values for the unknown quantity. These values will both satisfy the algebraic conditions, and sometimes they will both satisfy the particular conditions of the problem, but in most cases but one value of the unknown is applicable to the problem; and the value to be used must be determined from the nature of the question.

We will illustrate this principle by the solution of some particular questions,

1. Find a number such that its square being subtracted from five times the number, shall give 6 for remainder.

Let $x =$ the number sought.

Then, by the conditions of the questions, we have

$$5x - x^2 = 6. \qquad (1)$$

Changing all the signs of (1), it becomes

$$x^2 - 5x = -6. \qquad (2)$$

which, when solved by the rule for quadratics, gives

$$x = \frac{5 \pm 1}{2} = 3, \text{ or } 2.$$

Taking the first value of $x = 3$, we find its square to be 9.

Five times this value of x, is $5 \times 3 = 15$.

And $15 - 9 = 6$; therefore the number 3 satisfies the question.

The number 2 will satisfy it equally well, since its square $= 4$, which, subtracted from five times $2 = 10$, gives for remainder 6.

2. Find a number such that when added to 6, and the sum multiplied by the number, the product will equal the number diminished by 6.

Let $x =$ the number sought; then, by the conditions of the question, we have

$$(x + 6)x = x - 6. \qquad (1)$$

Expanding and collecting terms, we find

$$x^2 + 5x = -6. \qquad (2)$$

This solved gives

$$x = \frac{-5 \pm 1}{2} = -3, \text{ or } -2.$$

Here, as in the last question, we find that both values of x will satisfy our question.

If we take the first value, $x = -3$, we find that the number -3 added to 6 gives 3, which multiplied by -3 gives -9; and this is the same as -3 diminished by 6.

If we take the second value, $x = -2$, we find that the number -2 added to 6 gives 4, which multiplied by -2 gives -8; and this is the same as -2 diminished by 6.

3. Find a number which subtracted from its square, shall give 6 for remainder.

Let $x =$ the number, then we have

$$x^2 - x = 6. \qquad (1)$$

This gives

$$x = \frac{1 \pm 5}{2} = 3, \text{ or } -2.$$

If we take 3 for the number, its square is 9, from which subtracting 3, we have 6.

Again, taking -2 for the number, its square is 4, from which subtracting -2, we have 6.

So that both values of x satisfy the conditions of the question.

4. A and B travel from the same place, and in the same direction. The first day A travels but 1 mile, the second day he goes 3 miles, the third day 5 miles, and so on in arithmetical progression. After A has been gone 8 days, B follows, travelling uniformly at the rate of 36 miles each day. How many days after B starts will they be together?

Let $x =$ the number of days sought.

Then will $x + 8 =$ the number of days which A travelled,

$(x+8) =$ distance travelled by A.

$36x =$ " " B.

Hence,

$$(x + 8)^2 = 36x.$$

This, solved by the usual method of quadratics, gives

$$x = 4, \text{ or } x = 16.$$

From which we learn that they were twice together. First B overtakes A at the end of 4 days, and then in 12 days more A overtakes B.

In this case both answers are applicable.

5. By selling a watch for \$24, I lose as much per cent as the watch cost me. What was the cost of the watch?

Let x = the number of dollars the watch cost.

Then will $x-24$ = the loss incurred by the sale.

The loss per cent. will be

$$\frac{(x-24)100}{x}.$$

Therefore, by the question, we have this equation :

$$\frac{100(x-24)}{x}=x.$$

This gives $x=40$, or $x=60$.

In this case, also, both answers are applicable.

6. There is a number consisting of two digits, of which the right-hand digit is 3 greater than the left-hand digit, and the number itself is equal to the square of the right-hand digit. What is the number?

Ans. 25, or 36.

7. A number consists of two digits, of which the right-hand digit is double the left-hand digit. The number exceeds the square of the right-hand digit by 8. What is the number? Ans. 12, or 24.

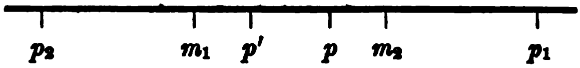
8. A and B speaking of their ages, A said he was 15 years older than B, and that the square of his age was equal to 64 times B's age. What were their ages?

$$\text{Ans. } \begin{cases} A=40 \text{ and } B=25\text{; or,} \\ A=24 \text{ and } B=\ \ 9. \end{cases}$$

9. *By the law of universal attraction we know, that the attraction of different bodies, at different distances, varies directly as their masses and inversely as the squares of their distances from the attracted point.*

The above law being admitted, it is required to find a point in the right line which joins the centres of the two spherical bodies, whose masses are m_1 and m_2, such that

this point will be attracted with equal force by each of the bodies.



Let m_1 and m_2 be the position of the bodies.

Let the distance between the centres of the two bodies m_1 and $m_2 = d$.

Also, let p denote the point sought.

Put $x = m_1p =$ the distance from the body m_1 to the point sought, measured from m_1 towards the right.

Then $d - x = m_2p =$ the distance from the body m_2 to the point, measured from m_2 towards the left.

Now, having reference to the above law, we know that the attractions of the two bodies upon the point p will be to each other as the expressions

$$\frac{m_1}{x^2}, \quad \frac{m_2}{(d-x)^2}.$$

But, by the questions, these forces of attraction are equal; therefore we have this condition:

$$\frac{m_1}{x^2} = \frac{m_2}{(d-x)^2}. \qquad (1)$$

Extracting the square root of both members of (1), we have

$$\frac{\sqrt{m_1}}{x} = \frac{\pm\sqrt{m_2}}{d-x}. \qquad (2)$$

This reduced, by rules for simple equations, gives

$$m_1p = x = \frac{\sqrt{m_1}}{\sqrt{m_1} \pm \sqrt{m_2}} \times d. \qquad (3)$$

Hence,

$$m_2p = d - x = \frac{\pm\sqrt{m_2}}{\sqrt{m_1} \pm \sqrt{m_2}} \times d. \qquad (4)$$

If we use the upper signs, we get

$$\left.\begin{aligned} m_1p &= \frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}}\times d, \\ m_2p &= \frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}\times d. \end{aligned}\right\} \qquad (A)$$

By taking the lower signs, we have

$$\left.\begin{aligned} m_1p &= \frac{\sqrt{m_1}}{\sqrt{m_1}-\sqrt{m_2}}\times d, \\ m_2p &= -\frac{\sqrt{m_2}}{\sqrt{m_1}-\sqrt{m_2}}\times d. \end{aligned}\right\} \qquad (B)$$

We will now interpret these expressions for different numerical values of d, m_1, m_2, and, in order that the following reasoning may be rigidly correct, it is necessary to suppose all the matter of the bodies m_1 and m_2 to be concentrated at the centres of the bodies.

CASE I.

When $d =$ a finite quantity.
And $m_1 > m_2$.

In this case, we evidently have

$$\frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}} > \tfrac{1}{2} \text{ and } < 1$$

$$\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}} < \tfrac{1}{2}.$$

Consequently, the first set of values, denoted by (A), give

$$m_1p = \text{a positive quantity which is } < d, \text{ but } > \frac{d}{2}.$$

$$m_2p = \text{a positive quantity which is } < \frac{d}{2}.$$

These values give for the point sought, a position between m_1 and m_2, but nearer m_2 than m_1.

Again,

$$\frac{\sqrt{m_1}}{\sqrt{m_1}-\sqrt{m_2}}>1.$$

$$-\frac{\sqrt{m_2}}{\sqrt{m_1}-\sqrt{m_2}}=\text{a negative quantity.}$$

Therefore, the second set of values, denoted by (B), give

$m_1p =$ a positive quantity which is $>d$.

$m_2p =$ a negative quantity.

Now, since the distances from m_2, measured towards the left, are considered as positive, the distances in an opposite direction must be regarded as negative.

Hence, these second values give for the point a position on the right of m_2.

CASE II.

When $d =$ a finite quantity.
And $m_1<m_2$.

In this case

$$\frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}}<\tfrac{1}{2}.$$

$$\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}>\tfrac{1}{2}\text{ and }<1.$$

Consequently the first set of values, denoted by (A), give

$m_1p =$ a positive quantity $>\frac{d}{2}$.

$m_2p =$ a positive quantity $>\frac{d}{2}$.

And the point lies between m_1 and m_2, nearer m_1 than m_2.

Again,

$$\frac{\sqrt{m_1}}{\sqrt{m_1}-\sqrt{m_2}}=\text{a negative quantity.}$$

$$-\frac{\sqrt{m_2}}{\sqrt{m_1}-\sqrt{m_2}}=\text{a positive quantity} > 1.$$

Therefore these second values give for the point a position on the left of m_1.

This case is obviously the same as Case I., when we interchange the bodies m_1 and m_2.

CASE III.

When $d =$ a finite quantity.
And $m_1 = m_2$.

In this case,

$$\frac{\sqrt{m_1}}{\sqrt{m_1}+\sqrt{m_2}}=\tfrac{1}{2}.$$

$$\frac{\sqrt{m_2}}{\sqrt{m_1}+\sqrt{m_2}}=\tfrac{1}{2}.$$

Consequently, the first set of values, denoted by (A), give

$$m_1p=\frac{d}{2}.$$

$$m_2p=\frac{d}{2}.$$

And the point is equi-distant from m_1 and m_2.

Again,

$$\frac{\sqrt{m_1}}{\sqrt{m_1}-\sqrt{m_2}}=\pm\frac{\sqrt{m_1}}{0}=\pm\text{an infinite quantity. (Art.133.)}$$

$$-\frac{\sqrt{m_2}}{\sqrt{m_1}-\sqrt{m_2}}=\mp\frac{\sqrt{m_2}}{0}=\mp\text{an infinite quantity. (Art.133.)}$$

Therefore, the second set of values, denoted by (B), give for the point a position at an infinite distance either to the right or left.

CASE IV.

When $d=0$. *And* $m_1 \gtrless m_2$.

In this case, we have

$$m_1p=0,$$
$$m_2p=0,$$

for both sets of values; consequently there is but one point which is equally attracted by both bodies, and that point is the common centre of the two bodies.

CASE V.

When $d=0$. *And* $m_1=m_2$.

The first set of values evidently become

$$m_1p=0,$$
$$m_2p=0.$$

Which shows that the point is in the common centre of the two bodies.

The second set of values give

$$m_1p=\frac{0}{0}=\text{an indeterminate quantity. (Art. 134.)}$$

$$m_2p=\frac{0}{0}=\text{an indeterminate quantity. (Art. 134.)}$$

So that the point may be any where on the line which joins the centres of the bodies. Since the two centres are united, every line which passes through this common point may be regarded as joining those centres; consequently every point

in space is, in this particular case, equally attracted by each body.

From the above discussion, we see that the analytical expressions are faithful to give all the particular cases which are possible to arise from giving particular values to the constant quantities which enter into the conditions of the question.

(162.) We will now add a couple examples for the purpose of illustrating the case in which the roots are imaginary.

1. Find two numbers whose sum is 8, and whose product is 17.

Let $x =$ one of the numbers, then will $8 - x =$ the other number.

The product is $(8 - x)x = 8x - x^2$, which, by the conditions of the question, is 17.

Therefore, we have this equation of condition,

$$x^2 - 8x = -17. \quad (1)$$

This, solved by the usual rules for quadratics, gives

$$x = 4 \pm \sqrt{-1}, \text{ for one of the numbers,}$$

and $8 - (4 \pm \sqrt{-1}) = (4 \mp \sqrt{-1},)$ for the other number.

Therefore, the numbers are $\begin{cases} 4 \pm \sqrt{-1}, \\ 4 \mp \sqrt{-1}, \end{cases}$

both of which are imaginary; we are therefore authorized to conclude that it is *impossible* to find two numbers whose sum is 8, and product 17.

We may also satisfy ourselves of this as follows: Since the sum of the two numbers is 8, they must average just 4; hence the greater must exceed 4 just as much as the less

falls short of 4. Therefore any two numbers whose sum is 8 may be represented by

$$4+x,$$
$$4-x.$$

Taking their product, we have

$$(4+x)(4-x)=16-x^2.$$

Now, since x^2 is positive for all *real* values of x, it follows that the product $16-x^2$ is always less than 16; that is, *no two real numbers whose sum is* 8, *can be found such that their product can equal* 17.

If we put the expression for the product, which we have just found equal to 17, we shall have

$$16-x^2=17,$$

$$\text{consequently,}\quad x=\pm\sqrt{-1}.$$

$$\text{And,}\ \left.\begin{aligned}4+x&=4\pm\sqrt{-1},\\ 4-x&=4\mp\sqrt{-1},\end{aligned}\right\}\text{the same values as found by the}$$

first method.

These values, although they are imaginary, will satisfy the algebraic conditions of the question; that is, their sum is

$$(4\pm\sqrt{-1})+(4\mp\sqrt{-1})=8,$$

and their product is

$$(4\pm\sqrt{-1})\times(4\mp\sqrt{-1})=17.$$

2. Find two numbers whose sum is 2, and sum of their reciprocals 1.

Denoting the numbers by x and y, we have the following relations:

$$\left.\begin{aligned}x+y&=2,\\ \frac{1}{x}+\frac{1}{y}&=1.\end{aligned}\right\}\qquad(1)$$

These, solved by the ordinary rules, give

$$\left.\begin{aligned} x &= 1 \pm \sqrt{-1}, \\ y &= 1 \mp \sqrt{-1}. \end{aligned}\right\} \qquad (2)$$

Both these values are imaginary; consequently the conditions of the question are absurd.

We may also show the impossibility of this question as follows: The sum being 2 the numbers may be denoted by

$$\left.\begin{aligned} 1+x, \\ 1-x. \end{aligned}\right\}$$

Taking the sum of their reciprocals, we have

$$\frac{1}{1+x}+\frac{1}{1-x},$$

which, when reduced to a common denominator, becomes

$$\frac{2}{1-x^2}.$$

The denominator of this expression cannot be greater than 1; for all real values of x, the expression must exceed 2. Therefore, *it is impossible to find two numbers whose sum shall equal* 2, *and sum of their reciprocals equal* 1.

(163.) From what has been said, we conclude that when, in the course of the solution of an algebraic problem, we fall upon imaginary quantities, there must be conditions in the problem which are incompatible.

Under Art. 128, we remarked that imaginary quantities had been advantageously employed as aids in the solution of many refined and delicate problems of the higher parts of analysis; here we notice their utility in pointing out the impossibility of questions, which otherwise, with only a superficial investigation, might be supposed possible.

CHAPTER VI.

RATIO AND PROGRESSION.

(164.) By *Ratio* of two quantities we mean their relation. When we compare quantities, by seeing how much greater one is than another, we obtain *arithmetical ratio.* Thus: the arithmetical ratio of 6 to 4 is 2, since 6 exceeds 4 by 2; in the same way, the arithmetical ratio of 11 to 7 is 4.

In the relation $a-c=r, \quad (1)$

r is the *arithmetical ratio* of a to c.

The first of the two terms which are compared is called the *antecedent*; the second is called the *consequent.* Thus, referring to (1), we have

$$a = \textit{antecedent}.$$
$$c = \textit{consequent}.$$
$$r = \textit{ratio}.$$

From (1), we get by transposition,

$$a=c+r, \quad (2)$$
$$c=a-r. \quad (3)$$

Equation (2) shows, *that in an arithmetical ratio the antecedent is equal to the consequent increased by the ratio.*

Equation (3) in like manner shows, *that the consequent is equal to the antecedent diminished by the ratio.*

(165.) When the arithmetical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form an *arithmetical proportion*.

Thus, if $a - c = r$; and $a' - c' = r$, then will

$$a - c = a' - c', \tag{4}$$

which relation is an arithmetical proportion, and is read thus:

a is as much greater than c, as a' is greater than c'.

Of the four quantities constituting an arithmetical proportion, the first and fourth are called the *extremes*, the second and third are called the *means*.

The first and second, together, constitute the *first couplet*; the third and fourth constitute the *second couplet*.

From equation (4), we get by transposing,

$$a + c' = a' + c, \tag{5}$$

which shows, *that the sum of the extremes, of an arithmetical proportion, is equal to the sum of the means.*

If $c = a'$, then (4) becomes

$$a - a' = a' - c', \tag{6}$$

which changes (5) into

$$a + c' = 2a'. \tag{7}$$

So that, if three terms constitute an arithmetical proportion, the sum of the extremes will equal twice the mean.

(166.) A series of quantities which increase or decrease by a constant difference form an *arithmetical progression*. When the series is increasing, it is called an *ascending progression*; when decreasing it is called a *descending progression*.

Thus, of the two series

$$1,\ 3,\ 5,\ 7,\ 9,\ 11,\ \&c. \tag{8}$$

$$27,\ 23,\ 19,\ 15,\ 11,\ 7,\ \&c. \tag{9}$$

The first is an ascending progression, whose ratio or *common difference* is 2; the second is a descending progression, whose *common difference* is 4.

(167.) If $a =$ the first term of an ascending arithmetical progression, whose common difference $= d$, the successive terms will be

$$\left.\begin{array}{r} a = \text{first term}, \\ a + d = \text{second term}, \\ a + 2d = \text{third term}, \\ a + 3d = \text{fourth term}, \\ \text{- - - - - - - -} \\ \text{- - - - - - - -} \\ a + (n-1)d = n\text{th term}. \end{array}\right\} \qquad (10)$$

If we denote the last or nth term by l, we shall have

$$l = a + (n-1)d. \qquad (11)$$

From (11) we readily deduce

$$a = l - (n-1)d, \qquad (12)$$

$$d = \frac{l-a}{n-1}, \qquad (13)$$

$$n = \frac{l-a}{d} + 1. \qquad (14)$$

When the progression is descending, we must write $-d$ for d in the above formulas.

Suppose, in an arithmetical progression, x to be a term which is preceded by q terms; and y to be a term which is followed by q terms; then by using (11) we have

$$x = a + qd, \qquad (15)$$

$$y = l - qd. \qquad (16)$$

Taking the sum of (15) and (16), we get

$$x + y = a + l. \qquad (17)$$

That is, the sum of any two terms equi-distant from the extremes is equal to the sum of the extremes, so that the terms will average half the sum of the extremes; consequently, *the sum of all the terms equals half the sum of the extremes multiplied by the number of terms.*

Representing the sum of n terms by s, we have

$$s=\frac{a+l}{2}\times n. \qquad (18)$$

From (18) we easily obtain

$$a=\frac{2s}{n}-l. \qquad (19)$$

$$l=\frac{2s}{n}-a. \qquad (20)$$

$$n=\frac{2s}{a+l}. \qquad (21)$$

Any three of the quantities

$a=$ the first term,
$d=$ common difference,
$n=$ number of terms,
$l=$ last term,
$s=$ sum of all the terms,

being given, the remaining two can be found, which must give rise to 20 different formulas, as given in the following table for ARITHMETICAL PROGRESSION.

(168.) We have not deemed it necessary to exhibit the particular process of finding each distinct formula of the following table, since they are all derived from the two fundamental ones, (1) and (7), by the usual operations upon equations not exceeding the second degree. It will furnish a good exercise for the student to deduce all these formulas by the aid, only, of formulas 1 and 7.

No.	Given.	Required.	Formulas.	Corr.
1	a, d, n	l	$l=a+(n-1)d$	17
2	a, d, s		$l=-\frac{1}{2}d\pm\sqrt{2ds+(a-\frac{1}{2}d)^2}$	19
3	a, n, s		$l=\frac{2s}{n}-a$	20
4	d, n, s		$l=\frac{s}{n}+\frac{(n-1)d}{2}$	18
5	a, d, n	s	$s=\frac{1}{2}n[2a+(n-1)d]$	8
6	a, d, l		$s=\frac{l+a}{2}+\frac{(l+a)(l-a)}{2d}$	
7	a, n, l		$s=\frac{1}{2}n(a+l)$	
8	d, n, l		$s=\frac{1}{2}n[2l-(n-1)d]$	5
9	a, n, l	d	$d=\frac{l-a}{n-1}$	
10	a, n, s		$d=\frac{2s-2an}{n(n-1)}$	12
11	a, l, s		$d=\frac{(l+a)(l-a)}{2s-l-a}$	
12	n, l, s		$d=\frac{2nl-2s}{n(n-1)}$	10
13	a, d, l	n	$n=\frac{l-a}{d}+1$	
14	a, d, s		$n=\frac{d-2a}{2d}\pm\sqrt{\frac{2s}{d}+\left(\frac{2a-d}{2d}\right)^2}$	16
15	a, l, s		$n=\frac{2s}{l+a}$	
16	d, l, s		$n=\frac{2l+d}{2d}\pm\sqrt{\left(\frac{2l+d}{2d}\right)^2-\frac{2s}{d}}$	14
17	d, n, l	a	$a=l-(n-1)d$	1
18	d, n, s		$a=\frac{s}{n}-\frac{(n-1)d}{2}$	4
19	d, l, s		$a=\frac{1}{2}d\pm\sqrt{(l+\frac{1}{2}d)^2-2ds}$	2
20	n, l, s		$a=\frac{2s}{n}-l$	3

(169.) From the nature of an arithmetical progression, we discover that if we subtract the common difference from the last term, we shall obtain the term next to the last; if we subtract from the last term twice the common difference, we obtain the second term from the last. Hence the terms of an arithmetical progression will be reversed if we interchange the values of a and l, and at the same time change the sign of d. Thus, the general form of an arithmetical progression is

$$a,\ a+d,\ a+2d, \ldots\ldots\ldots l-2d,\ l-d,\ l.$$

Changing a to l, l to a, and changing the sign of d, we have

$$l,\ l-d,\ l-2d, \ldots\ldots\ldots a+2d,\ a+d,\ a,$$

which is precisely the same progression as the first, with the terms arranged in a reverse order. The above change has, of course, no effect upon the number of terms, nor upon the sum of all the terms.

Therefore, in any of the formulas of the preceding table we are at liberty to make the above named changes. As an example, we will take from the table formula 2, which is

$$l = -\tfrac{1}{2}d \pm \sqrt{2ds+(a-\tfrac{1}{2}d)^2}.$$

Now, changing l to a, a to l, and changing the sign of d, it becomes

$$a = \tfrac{1}{2}d \pm \sqrt{(l+\tfrac{1}{2}d)^2-2ds},$$

which is formula 19.

In the same way, formulas 14 and 16 may be deduced from each other. Such formulas as may be derived from each other by the above changes we shall call correlative formulas. It is evident that some of the formulas of the table have no correlative. Thus, formulas 13 and 15 are not altered by the above changes. Those formulas which have correlative formulas have them referred to in the table, under column headed *Corr*.

EXAMPLES.

1. The first term of an arithmetical progression is 7, the common difference is $\frac{1}{4}$, and the number of terms is 16. What is the last term?

To solve this, we take formula 1 from our table, which is

$$l = a + (n - 1)d.$$

Substituting the above given values for a, d, and n, we find

$$l = 7 + \tfrac{1}{4}(16 - 1) = 10\tfrac{3}{4}.$$

2. The first term of an arithmetical progression is $\frac{3}{4}$, the common difference is $\frac{1}{8}$, and the last term is $3\frac{7}{8}$. What is the number of terms?

In this example we take formula 13.

$$n = \frac{l - a}{d} + 1\,;$$

which in this present case becomes

$$n = \frac{3\frac{7}{8} - \frac{3}{4}}{\frac{1}{8}} + 1 = 26.$$

3. One hundred stones being placed on the ground in a straight line, at the distance of two yards from each other, how far will a person travel who shall bring them one by one to a basket, placed at two yards from the first stone?

In this example $a = 4$; $d = 4$; $n = 100$; which values being substituted in formula 5, give

$$s = 50\{8 + 99 \times 4\} = 20200 \text{ yards},$$

which, divided by 1760, the number of yards in one mile, we get

$$s = 11 \text{ miles, } 840 \text{ yards.}$$

4. What is the sum of n terms of the progression

$$1, 3, 5, 7, 9, \ldots\ldots ?$$

Ans. $s = n^2$.

5. What is the sum of n terms of the progression

$$1, 2, 3, 4, 5, \ \ldots\ldots ?$$

Ans. $s=\frac{n(n+1)}{2}$.

GEOMETRICAL RATIO.

(170.) When we compare quantities by seeing how many times greater one is than another, we obtain *geometrical ratio*. Thus the geometrical ratio of 8 to 4 is 2, since 8 is 2 times as great as 4. Again, the geometrical ratio of 15 to 3 is 5.

In the relation, $$\frac{a}{c}=r, \qquad (1)$$

r is the geometrical ratio of a to c.

As in arithmetical ratio,

$a=$ *antecedent*,
$c=$ *consequent*,
$r=$ *ratio*.

From (1), we get by reduction,

$$a=cr, \qquad (2)$$

$$c=\frac{a}{r}. \qquad (3)$$

Equation (2) shows, *that in a geometrical ratio the antecedent is equal to the consequent multiplied by the ratio.*

Equation (3) shows, *that the consequent is equal to the antecedent divided by the ratio.*

(171.) When the geometrical ratio of any two terms is the same as the ratio of any other two terms, the four terms together form a *geometrical proportion*.

Thus, if $\frac{a}{c}=r$; and $\frac{a'}{c'}=r$, then will

$$\frac{a}{c}=\frac{a'}{c'}, \qquad (4)$$

which relation is a geometrical proportion, and is generally written thus:

$$a \; : \; c \; : \; : \; a' \; : \; c', \qquad (5)$$

which is read as follows: *a is to c, as a′ is to c′.*

Of the four quantities which constitute a geometrical proportion, as in arithmetical proportion, the first and fourth are called the *extremes*, the second and third are called the *means*.

The first and second constitute the *first couplet;* the third and fourth constitute the *second couplet.*

From equation (5), or its equivalent (4), we find

$$ac' = a'c, \qquad (6)$$

which shows, *that the product of the extremes of a geometrical proportion, is equal to the product of the means.*

If $c = a'$, then (5) becomes

$$a \; : \; a' \; : \; : \; a' \; : \; c', \qquad (7)$$

which changes (6) into

$$ac' = a'^2, \qquad (8)$$

so that if the two means which constitute a geometrical proportion be equal, then the product of the extremes will equal the square of the mean.

(172.) Quantities are said to be in proportion *by inversion*, or *inversely*, when the consequents are taken as antecedents, and the antecedents as consequents.

From (5), or its equivalent (4), which is

$$\frac{a}{c} = \frac{a'}{c'}, \qquad (9)$$

we have, by inverting both terms,

$$\frac{c}{a} = \frac{c'}{a'}.$$

Therefore, by Art. 171,

$$c \; : \; a \; : \; : \; c' \; : \; a'. \qquad (10)$$

Which shows, that if four quantities are in proportion they will be in proportion by inversion.

(173.) Quantities are in proportion by *alternation*, or *alternately*, when the antecedents form one of the couplets, and the consequents form the other.

Resuming (4),

$$\frac{a}{c}=\frac{a'}{c'}. \qquad (11)$$

Multiplying both terms of (11) by $\frac{c}{a'}$, it will become

$$\frac{a}{a'}=\frac{c}{c'}.$$

Therefore, by Art.171,

$$a : a' : : c : c'. \qquad (12)$$

Which shows, that if four quantities are in proportion they will be so by alternation.

(174.) Quantities are in proportion by *composition*, when the sum of the antecedent and consequent is compared either with antecedent or consequent.

Resuming (4),

$$\frac{a}{c}=\frac{a'}{c'}. \qquad (13)$$

If to (13) we add the terms of the following equation $\frac{c}{c}=\frac{c'}{c'}$, each of whose members is equal to unity, we have

$$\frac{a+c}{c}=\frac{a'+c'}{c'}.$$

Therefore, by Art. 171,

$$a+c : c : : a'+c' : c'. \qquad (14)$$

Which shows, that if four quantities are in proportion they will be so by composition.

(175.) Quantities are said to be in proportion by *division*, when the difference of antecedent and consequent is compared with either antecedent or consequent.

If we subtract the equation $\frac{c}{c}=\frac{c'}{c'}$, each member of which is equal to 1, from equation (4), we find

$$\frac{a-c}{c}=\frac{a'-c'}{c'}.$$

Therefore, by Art. 171, we have

$$a-c : c : : a'-c' : c'. \qquad (15)$$

Which shows, that if four quantities are in proportion, they will be so by division.

Equation (4) is

$$\frac{a}{c}=\frac{a'}{c'}.$$

Raising each member to the nth power, we have

$$\frac{a^n}{c^n}=\frac{a'^n}{c'^n}.$$

Therefore, by Art. 171, we have

$$a^n : c^n : : a'^n : c'^n. \qquad (16)$$

Which shows, that if four quantities are in proportion, like powers or roots of these quantities will also be in proportion.

If we have
$$\left.\begin{array}{l} a : c : : a' : c', \\ a : c : : a'' : c'', \\ a : c : : a''' : c''', \end{array}\right\} \qquad (17)$$
$$\&c., \qquad \&c.$$

These give by alternation, Art. 173,

$$\begin{array}{l} a : a' : : c : c', \\ a : a'' : : c : c'', \\ a : a''' : : c : c''', \\ \&c., \qquad \&c. \end{array}$$

Therefore, by inversion, Art. 172, we have

$$\left.\begin{array}{l}\dfrac{a'}{a}=\dfrac{c'}{c},\\[2ex] \dfrac{a''}{a}=\dfrac{c''}{c},\\[2ex] \dfrac{a'''}{a}=\dfrac{c'''}{c},\\[1ex] \&c., \&c.\\[1ex] \dfrac{a}{a}=\dfrac{c}{c}.\end{array}\right\} \qquad (18)$$

We also have

Taking the sum of equations (18), we have

$$\frac{a+a'+a''+a'''+\&c.}{a}=\frac{c+c'+c''+c'''+\&c.}{c}. \quad (19)$$

Therefore, by Art. 171, we have

$$a+a'+a''+a'''+\&c.: a : : c+c'+c''+c'''+\&c.: c. \quad (20)$$

Which shows, that if any number of quantities are proportional, the sum of all the antecedents will be to any one antecedent, as the sum of all the consequents is to its corresponding consequent.

(176.) If we have

$$a : c : : a' : c',$$
$$a'' : c'' : : a''' : c''',$$

then we find

$$\frac{a}{c}=\frac{a'}{c'}, \quad (21)$$

$$\frac{a''}{c''}=\frac{a'''}{c'''}. \quad (22)$$

Multiplying together the equations (21) and (22), we have

$$\frac{a\times a''}{c\times c''}=\frac{a'\times a'''}{c'\times c'''}. \quad (23)$$

Therefore, by Art. 171, we have

$$a \times a'' : c \times c'' : : a' \times a''' : c' \times c'''. \qquad (24)$$

Which shows, that if there be two sets of proportional quantities, the products of the corresponding terms will be proportional.

(177.) A series of quantities which increase or decrease by a constant multiplier forms a *geometrical progression.* When the series is increasing, that is, when the constant multiplier exceeds a unit, it is called an *ascending progression;* when decreasing, or when the constant multiplier is less than a unit, then it is called a *descending progression.*

Thus, of the two series,

$$1,\ 3,\ 9,\ 27,\ 81,\ 243,\ \&c., \qquad (25)$$
$$256,\ 128,\ 64,\ 32,\ 16,\ 8,\ \&c. \qquad (26)$$

the first is an ascending progression, whose constant multiplier or *ratio* is 3; the second is a descending progression, whose ratio is $\frac{1}{2}$.

(178.) If a is the first term of a geometrical progression, whose ratio $= r$, the successive terms will be

$$\left.\begin{array}{r} a = \text{first term,} \\ ar = \text{second term,} \\ ar^2 = \text{third term,} \\ ar^3 = \text{fourth term,} \\ - - - - - - - \\ - - - - - - - \\ ar^{n-1} = n\text{th term.} \end{array}\right\} \qquad (27)$$

If we denote the last or nth term by l, we shall have

$$l = ar^{n-1}. \qquad (28)$$

If we represent the sum of n terms of a geometrical progression by s, we shall have

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}. \qquad (29)$$

Multiplying all the terms of (29) by the ratio r, we have

$$rs = ar + ar^2 + ar^3 + ar^4 + \dots + ar^{n-1} + ar^n. \qquad (30)$$

Subtracting (29) from (30), we get

$$(r-1)s = a(r^n - 1). \qquad (31)$$

Therefore, $$s = a \cdot \left\{ \frac{r^n - 1}{r - 1} \right\}. \qquad (32)$$

Any three of the quantities

$a =$ first term,
$r =$ ratio,
$n =$ number of terms,
$l =$ last term,
$s =$ sum of all the terms,

being given, the remaining two can be found, which as in arithmetical progression, must give rise to 20 different formulas, as given in the following table for GEOMETRICAL PROGRESSION.

No.	Given.	Required.	Formulas.	Cor.
1	a, r, n,	l	$l = ar^{n-1}$	9
2	a, r, s,		$l = \dfrac{a + (r-1)s}{r}$	11
3	a, n, s,		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0$	12
4	r, n, s,		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}$	10
5	a, r, n,	s	$s = \dfrac{a(r^n - 1)}{r - 1}$	8
6	a, r, l,		$s = \dfrac{rl - a}{r - 1}$	
7	a, n, l,		$s = \dfrac{l^{\frac{n}{n-1}} - a^{\frac{n}{n-1}}}{l^{\frac{1}{n-1}} - a^{\frac{1}{n-1}}}$	
8	r, n, l,		$s = \dfrac{l(r^n - 1)}{(r-1)r^{n-1}}$	5

No.	Given.			Requi-red.	Formulas.	Cor.
9	r,	n,	l,		$a=\frac{l}{r^{n-1}}$	1
10	r,	n,	s,	a	$a=\frac{(r-1)s}{r^n-1}$	4
11	r,	l,	s,		$a=rl-(r-1)s$	2
12	n,	l,	s,		$a(s-a)^{n-1}-l(s-l)^{n-1}=0$	3
13	a,	n,	l,		$r=\left(\frac{l}{a}\right)^{\frac{1}{n-1}}$	
14	a,	n,	s,	r	$r^n-\frac{s}{a}r+\frac{s-a}{a}=0$	16
15	a,	l,	s,		$r=\frac{s-a}{s-l}$	
16	n,	l,	s,		$r^n-\frac{s}{s-l}r^{n-1}+\frac{l}{s-l}=0$	14
17	a,	r,	l,		$n=\frac{\log l-\log a}{\log r}+1$	
18	a,	r,	s,	n	$n=\frac{\log[a+(r-1)s]-\log a}{\log r}$	20
19	a,	l,	s,		$n=\frac{\log l-\log a}{\log(s-a)-\log(s-l)}+1$	
20	r,	l,	s,		$n=\frac{\log l-\log[rl-(r-1)s]}{\log r}+1$	18

(179.) All the formulas of the above table are easily drawn from the conditions of (28) and (32), which conditions correspond with formulas (1) and (5), except the last four which involve logarithms; we will hereafter, under Logarithms, show how these formulas are obtained.

If in a geometrical progression we change a to l, l to a, and r to $r^{-1}=\frac{1}{r}$, the progression will remain the same as before, taken in a reverse order. These changes being made in the formulas of the preceding table, we shall discover that some of the formulas, as in arithmetical progression, have correlative formulas. Those having correlative formulas, have them referred to in the table, under column headed *Cor.*

EXAMPLES.

1. The first term of a geometrical progression is 5, the ratio 4, the number of terms is 9. What is the last term?

Formula (1), which is $l=ar^{n-1}$, gives

$$l=5\times 4^8=327680.$$

2. The first term of a geometrical progression is 4, the ratio is 3, the number of terms is 10. What is the sum of all the terms?

Formula (5), which is $s=\frac{a(r^n-1)}{r-1}$, gives

$$s=\frac{4(3^{10}-1)}{2}=118096.$$

3. The last term of a geometrical progression is $106\frac{403}{512}$ the ratio is $\frac{3}{2}$, the number of terms 8. What is the first term?

Formula (9), which is $a=\frac{l}{r^{n-1}}$, gives

$$a=\frac{106\frac{403}{512}}{(\frac{3}{2})^7}=6\tfrac{1}{4}.$$

(180.) When the progression is descending the ratio is less than one, and if we suppose the series extended to an infinite number of terms, the last term may be taken $l=0$, which causes formula 6 to become

$$s = \frac{a}{1-r}. \qquad (33)$$

Which shows, that the sum of an infinite number of terms of a descending geometrical progression is equal to its first term, divided by one diminished by the ratio.

EXAMPLES.

1. What is the sum of the infinite progression

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{16} + \&c.\ ?$$

In this example $a = 1$, $r = \frac{1}{2}$, and (33) becomes

$$s = \frac{1}{1-\frac{1}{2}} = 2.$$

2. What is the value of 0.33333 &c., or which is the same thing, of the infinite series $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \&c.$?

Here $a = \frac{3}{10}$, $r = \frac{1}{10}$, and (33) gives

$$s = \frac{\frac{3}{10}}{1-\frac{1}{10}} = \tfrac{3}{9} = \tfrac{1}{3}.$$

3. What is the value of 0.12121212 &c., or which is the same, of $\frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000}$ &c. ?

In this example $a = \frac{12}{100}$, $r = \frac{1}{100}$, and (33) gives

$$s = \frac{\frac{12}{100}}{1-\frac{1}{100}} = \tfrac{12}{99} = \tfrac{4}{33}.$$

4. What is the sum of the infinite series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \tfrac{1}{81} + \&c.\ ?$$

Ans. $\frac{3}{2}$.

5. What is the sum of the infinite series.

$$\tfrac{1}{5} + \tfrac{1}{25} + \tfrac{1}{125} + \&c.\ ?$$

Ans. $\frac{1}{4}$.

6. What is the sum of the series $1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+$ &c., to infinity?

Ans. $\frac{x}{x-1}$.

7. What is the sum of the series $1+\frac{1}{x+1}+\frac{1}{(x+1)^2}$ $+\frac{1}{(x+1)^3}+$ &c., to infinity?

Ans. $\frac{x+1}{x}$.

8. Suppose the elastic power of a ball, which falls from a height of 100 feet, to be such as to cause it to rise 0.9375 of the height from which it fell; and to continue in this way diminishing the height to which it will rise in geometrical progression, till it comes to rest. How far will it have moved?

Ans. 3260 feet.

HARMONICAL PROPORTION.

(181.) Three quantities are in harmonical proportion, when the first has the same ratio to the third, as the difference between the first and second has to the difference between the second and third.

Four quantities are in harmonical proportion, when the first has the same ratio to the fourth, as the difference between the first and second has to the difference between the third and fourth.

Thus, if

$$a : c : : a-b : b-c, \qquad (1)$$

then will the three quantities a, b, c, be in harmonical proportion.

If $$a : d : : a-b : c-d, \qquad (2)$$

then also will the four quantities a, b, c, and d be in harmonical proportion.

Multiplying means and extremes of (1), we have

$$ab - ac = ac - bc, \tag{3}$$

which by transposition becomes

$$ab + bc = 2ac. \tag{4}$$

In a similar way equation (2) gives

$$ac + bd = 2ad. \tag{5}$$

Suppose a, b, c, d, e, &c., to be in harmonical progression; then from (4) we have

$$\left.\begin{array}{l} bc + ab = 2ac, \\ cd + bc = 2bd, \\ de + cd = 2ce, \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\} \tag{6}$$

Dividing the first of (6) by abc, the second by bcd, and the third by cde, &c., we find

$$\left.\begin{array}{l} \frac{1}{a} + \frac{1}{c} = \frac{2}{b}, \\ \frac{1}{b} + \frac{1}{d} = \frac{2}{c}, \\ \frac{1}{c} + \frac{1}{e} = \frac{2}{d}, \\ \text{\&c.} \quad \text{\&c.} \end{array}\right\} \tag{7}$$

From which we see that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \frac{1}{e}$, &c., are in arithmetical progression. (Art. 165.)

Hence, the reciprocals of any number of terms in harmonical progression are in arithmetical progression; and conversely the reciprocals of the terms of any arithmetical progression must be in harmonical progression.

The reciprocals of the arithmetical series 1, 2, 3, 4, 5, 6, are $\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$, whose numerators, when reduced to a common denominator, are 60, 30, 20, 15, 12, 10, which by the above property must be in harmonical progression.

If six musical strings of equal tension and thickness, have their lengths in proportion to the above numbers, they will, when sounded together, produce more perfect harmony than could be produced by strings of different lengths; and hence we see the propriety of calling this kind of relation, *harmonical* or *musical* proportion.

(182.) If we take the arithmetical mean, the geometrical mean, and the harmonical mean, of any two numbers, these three means will be in geometrical proportion.

Let a and b be any two numbers, then will

$$\tfrac{1}{2}(a+b) = \text{their arithmetical mean,}$$
$$\sqrt{ab} = \text{“ geometrical “}$$
$$\frac{2ab}{a+b} = \text{“ harmonical “}$$

And we evidently have

$$\tfrac{1}{2}(a+b) : \sqrt{ab} : : \sqrt{ab} : \frac{2ab}{a+b}.$$

That is,

The geometrical mean, between the arithmetical mean and the harmonical mean of two quantities, is the same as the geometrical mean of the quantities themselves.

CHAPTER VII.

SERIES.

METHOD OF INDETERMINATE COEFFICIENTS.

(183.) Suppose we have the following condition:

$$\left.\begin{array}{l} A_0+A_1x+A_2x^2+A_3x^3+\text{ \&c.} \\ =B_0+B_1x+B_2x^2+B_3x^3+\text{ \&c.} \end{array}\right\} \quad (1)$$

If the above condition is true for all values of x, we must have

$$\left.\begin{array}{l} A_0=B_0, \\ A_1=B_1, \\ A_2=B_2, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ A_n=B_n. \end{array}\right\} \quad (2)$$

For, since the condition (1) is true for all values of x, it becomes, when $x=0$, $A_0=B_0$.

Now, rejecting A_0 from the left-hand member of (1) and its equal B_0 from its right-hand member, it will become

$$A_1x+A_2x^2+A_3x^3+\text{ \&c.}=B_1x+B_2x^2+B_3x^3+\text{ \&c.} \quad (3)$$

Dividing through by x, we find

$$A_1+A_2x+A_3x^2+\text{ \&c.}=B_1+B_2x+B_3x^2+\text{\&c.} \quad (4)$$

When $x=0$, equation (4) becomes $A_1=B_1$.

By a similar process we can show, that $A_2 = B_2$; $A_3 = B_3$; and, in general, $A_n = B_n$.

If we transpose all the terms of the right-hand member of (1), it will become

$$\left.\begin{array}{r} A_0 - B_0 + (A_1 - B_1)x + (A_2 - B_2)x^2 \\ +(A_3 - B_3)x^3 + \&c. = 0. \end{array}\right\} \qquad (5)$$

(184.) *Hence, when we have an equation of the form of* (5), *true for all values of x, it follows, that the coefficients of the different powers of x, are respectively equal to* 0.

We will now apply the above principle in the development of some particular

EXAMPLES.

1. Required to expand $\frac{1+2x}{1-x-x^2}$ into an infinite series.

Assume,

$$\frac{1+2x}{1-x-x^2} = A_0 + A_1x + A_2x^2 + A_3x^3 + \&c.$$

Clearing this of fractions and then transposing, it becomes

$$\left.\begin{array}{r} A_0 + A_1 \\ 1 - A_0 \\ -2 \end{array}\right\} x \left.\begin{array}{r} +A_2 \\ -A_1 \\ -A_0 \end{array}\right\} x^2 \left.\begin{array}{r} +A_3 \\ -A_2 \\ -A_1 \end{array}\right\} x^3 + \&c. = 0.$$

Now, since the right-hand member is equal to 0, it fol lows, by the above principle of indeterminate coefficients that the coefficients of the left-hand member must each equal 0; hence we have the following conditions:

$$\left.\begin{array}{rr} A_0 - 1 = 0, & (1) \\ A_1 - A_0 - 2 = 0, & (2) \\ A_2 - A_1 - A_0 = 0, & (3) \\ A_3 - A_2 - A_1 = 0, & (4) \\ \dots\dots\dots\dots & \\ \dots\dots\dots\dots & \\ A_n - A_{n-1} - A_{n-2} = 0. & (n+1) \end{array}\right\} \qquad (A)$$

From the above we readily find

$$\left.\begin{array}{ll} A_0=1, & (1) \\ A_1=3, & (2) \\ A_2=4, & (3) \\ A_3=7, & (4) \\ \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots & \\ A_n=A_{n-1}+A_{n-2}, & (n+1) \end{array}\right\} \quad (B)$$

The value of the general coefficient A_n, as given in group (B), shows that, *any coefficient is equal to the sum of the two preceding ones.*

Substituting these values, as given by (B), in the assumed value of $\frac{1+2x}{1-x-x^2}$, we find

$$\frac{1+2x}{1-x-x^2}=1+3x+4x^2+7x^3+11x^4+18x^5+\ \&c.$$

2. Required the development of $\frac{x}{1+x+x^2}$ by this method.

Assume

$$\frac{x}{1+x+x^2}=A_0+A_1x+A_2x^2+A_3x^3+A_4x^4+\ \&c.,$$

proceeding as in last example, we find

$$\left.\begin{array}{r} A_0+A_1 \\ +A \\ -1 \end{array}\right\} x + \left.\begin{array}{r} +A_2 \\ +A_1 \\ +A_0 \end{array}\right\} x^2 + \left.\begin{array}{r} +A_3 \\ +A_2 \\ +A_1 \end{array}\right\} x^3 + \left.\begin{array}{r} +A_4 \\ +A_3 \\ +A_2 \end{array}\right\} x^4+\ \&c.=0.$$

Equating the coefficients to zero, we have

$$\left.\begin{array}{rr} A_0 = 0, & (1) \\ A_1 + A_0 - 1 = 0, & (2) \\ A_2 + A_1 + A_0 = 0, & (3) \\ A_3 + A_2 + A_1 = 0, & (4) \\ A_4 + A_3 + A_2 = 0, & (5) \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ A_n + A_{n-1} + A_{n-2} = 0. & (n+1) \end{array}\right\} \quad (B)$$

Commencing with the first condition, we find $A_0 = 0$, which substituted in (2) gives $A_1 = 1$, these values of A_0 and A_1, substituted in (3), give $A_2 = -1$, now substituting A_1 and A_2 in (4), we find $A_3 = 0$, continuing in this way, we find $A_4 = 1$; $A_5 = -1$, and so on; from the general condition $(n+1)$ we find $A_n = -A_{n-1} - A_{n-2}$, that is, *any coefficient is equal to the sum of the two preceding coefficients taken with a contrary sign.*

Hence, $\dfrac{x}{1+x+x^2} = x - x^2 + x^4 - x^5 + x^7 - \&c.$

3. Required the development of $\sqrt{1-x}$ by this method.

Assume, $\sqrt{1-x} = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \&c.$

Squaring both members, we find

$$1 - x = A_0^2 + 2A_0A_1 \left.\right\} x \begin{array}{c} +2A_0A_2 \\ +A_1^2 \end{array}\left.\right\} x^2 \begin{array}{c} +2A_0A_3 \\ +2A_1A_2 \end{array}\left.\right\} x^3 + \&c.$$

Equating like coefficients, we have

$$\left.\begin{array}{rr} A_0^2 = 1, & (1) \\ 2A_0A_1 = -1, & (2) \\ 2A_0A_2 + A_1^2 = 0, & (3) \\ 2A_0A_3 + 2A_1A_2 = 0, & (4) \\ 2A_0A_4 + 2A_1A_3 + A_2^2 = 0. & (5) \\ \cdots\cdots\cdots\cdots\cdots & \end{array}\right\} \quad (C)$$

The first condition gives

$$A_0=\sqrt{1}=1.$$

This value of A_0 substituted (2), we find

$$A_1=-\frac{1}{2}.$$

In this way we find, in succession, the following values :

$$A_2=-\frac{1}{2.4};\ A_3=-\frac{3}{2.4.6};\ A_4=-\frac{3.5}{2.4.6.8}\quad \&c.$$

These values substituted in our assumed value, give

$$\sqrt{1-x}=1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\&c.$$

The general term of this series is

$$-\frac{3.5\ldots(2n-3)x^n}{2.4.6.8\ldots.2n}.$$

4. Required the development of $\frac{1+x}{1-2x+x^2}$ by this method.

Ans. $1+3x+5x^2+7x^3+9x^4+11x^5+$ &c.

5. Required the development of $\frac{1+x}{1-x-x^2}$.

Ans. $1+2x+3x^2+5x^3+8x^4+13x^5+$ &c.

(185) Before closing this subject we will develop $\frac{x^m-y^m}{x-y}$, which will be of use hereafter.

Assume,

$$\frac{x^m-y^m}{x-y}=A_0+A_1y+A_2y^2+\ldots\ldots+A_my^m+\&c.\quad (1)$$

Multiplying through by $x-y$, we obtain

$$x^m - y^m = A_0 x + \left.\begin{matrix} A_1 x \\ -A_0 \end{matrix}\right\} y + \left.\begin{matrix} A_2 x \\ -A_1 \end{matrix}\right\} y^2 \ldots + \left.\begin{matrix} A_m x \\ -A_{m-1} \end{matrix}\right\} y^m + \text{\&c.} \quad (2)$$

Equating like coefficients of y, we get

$$\left.\begin{array}{lr} A_0 x = x^m & (1) \\ A_1 x - A_0 = 0, & (2) \\ A_2 x - A_1 = 0, & (3) \\ A_3 x - A_2 = 0, & (4) \end{array}\right\} \quad (A)$$

and in general,

$$A_n x - A_{n-1} = 0. \qquad (n+1)$$

Equating the coefficients of y^m, we have

$$A_m x - A_{m-1} = -1. \qquad (5)$$

From (1), we find

$$A_0 = x^{m-1},$$

which substituted in (2), we find

$$A_1 = x^{m-2}.$$

This in turn, substituted in (3), gives

$$A_2 = x^{m-3},$$

and in general we have

$$A_n = x^{m-n-1}.$$

In this general value of A_n write $m-1$ for n, and we get

$$A_{m-1} = x^0 = 1.$$

This value substituted in (5), gives

$$A_m x - 1 = -1, \text{ or } A_m = 0,$$

and consequently all the succeeding values of A_n will be reduced to zero.

These values of A_0; A_1; A_2; A_3; &c., substituted in (1), give

$$\frac{x^m - y^m}{x-y} = x^{m-1} + x^{m-2}y + x^{m-3}y^2 \ldots + xy^{m-2} + y^{m-1}. \quad (B)$$

BINOMIAL THEOREM.

(186.) We have already found by actual multiplication (Art. 94), that

$$\left.\begin{array}{l}(a+x)^1=a+x,\\(a+x)^2=a^2+2ax+x^2,\\(a+x)^3=a^3+3a^2x+3ax^2+x^3,\\(a+x)^4=a^4+4a^3x+6a^2x^2+4ax^3+x^4.\end{array}\right\}\qquad(A)$$

Now, the BINOMIAL THEOREM teaches us the law by which we may write the development of $(a+x)^m$ for any values of a, x, and m.

To determine this law, assume

$$(a+x)^{\frac{m}{n}}=A_0+A_1x+A_2x^2+A_3x^3+\&c.\qquad(1)$$

We have taken the exponent of this binomial fractional, in order to make the development more general.

The assumed form for the development of $(a+x)^{\frac{m}{n}}$ being general, must be true for all values of x. When $x=0$, it becomes $a^{\frac{m}{n}}=A_0$, introducing this value of A_0 in (1), we have

$$(a+x)^{\frac{m}{n}}=a^{\frac{m}{n}}+A_1x+A_2x^2+A_3x^3+\&c.\qquad(2)$$

In (2), writing x_1 for x, and it becomes

$$(a+x_1)^{\frac{m}{n}}=a^{\frac{m}{n}}+A_1x_1+A_2x_1^2+A_3x_1^3+\&c.\qquad(3)$$

Subtracting (3) from (2), we find

$$\left.\begin{array}{l}(a+x)^{\frac{m}{n}}-(a+x_1)^{\frac{m}{n}}=\\A_1(x-x_1)+A_2(x^2-x_1^2)+A_3(x^3-x_1^3)+\&c.\end{array}\right\}\qquad(4)$$

If we suppose

$$u=(a+x)^{\frac{1}{n}};\ u_1=(a+x_1)^{\frac{1}{n}},\qquad(5)$$

we readily find

$$u^m - u_1^m = (a+x)^{\frac{m}{n}} - (a+x_1)^{\frac{m}{n}}, \tag{6}$$

and

$$u^n - u_1^n = x - x_1. \tag{7}$$

Dividing the left-hand member of (4) by $u^n - u_1^n$, and the right-hand member by its equal $x - x_1$, observing to substitute $u^m - u_1^m$ for $(a+x)^{\frac{m}{n}} - (a+x_1)^{\frac{m}{n}}$, as given by (6), and it will become

$$\left.\begin{array}{l} \dfrac{u^m - u_1^m}{u^n - u_1^n} = \\ A_1\left(\dfrac{x - x_1}{x - x_1}\right) + A_2\left(\dfrac{x^2 - x_1^2}{x - x_1}\right) + A_3\left(\dfrac{x^3 - x_1^3}{x - x_1}\right) + \&c. \end{array}\right\} \tag{8}$$

Dividing both numerator and denominator of the left-hand member of (8) by $u - u_1$, and performing the divisions indicated in the right-hand member, and we obtain by the aid of equation (B), Art. 185, the following:

$$\left.\begin{array}{l} \dfrac{u^{m-1} + u_1 u^{m-2} + \ldots\ldots\ldots\ldots u_1^{m-1}}{u^{n-1} + u_1 u^{n-2} + \ldots\ldots\ldots\ldots u_1^{n-1}} = \\ A_1 + A_2(x + x_1) + A_3(x^2 + xx_1 + x_1^2) + \&c. \end{array}\right\} \tag{9}$$

Now, in (9), suppose $x = x_1$, and consequently $u = u_1$, and it becomes

$$\frac{mu^{m-1}}{nu^{n-1}} = \frac{mu^m}{nu^n} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \&c. \tag{10}$$

Re-substituting $(a+x)^{\frac{1}{n}}$ for u in (10), and it will become

$$\frac{m}{n} \cdot \frac{(a+x)^{\frac{m}{n}}}{a+x} = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + \&c. \tag{11}$$

Multiplying through by $a + x$, and we obtain

$$\left.\begin{array}{l}\frac{m}{n}(a+x)^{\frac{m}{n}} = \\ A_1a + \left.\begin{array}{l}2A_2a\\+A_1\end{array}\right\}x \left.\begin{array}{l}+3A_3a\\+2A_2\end{array}\right\}x^2 \left.\begin{array}{l}+4A_4a\\+3A_3\end{array}\right\}x^3 + \&c.\end{array}\right\} \quad (12)$$

Multiplying both members of (2) by $\frac{m}{n}$ and it becomes

$$\frac{m}{n}(a+x)^{\frac{m}{n}} = \frac{m}{n}a^{\frac{m}{n}} + \frac{m}{n}A_1x + \frac{m}{n}A_2x^2 + \&c. \quad (13)$$

Equating the right-hand members of (13), and (12), we have

$$\left.\begin{array}{l}\frac{m}{n}a^{\frac{m}{n}} + \frac{m}{n}A_1x + \frac{m}{n}A_2x^2 + \frac{m}{n}A_3x^3 + \&c. \\ = A_1a + \left.\begin{array}{l}2A_2a\\+A_1\end{array}\right\}x \left.\begin{array}{l}+3A_3a\\+2A_2\end{array}\right\}x^2 \left.\begin{array}{l}+4A_4a\\+3A_3\end{array}\right\}x^3 + \&c.\end{array}\right\} \quad (14)$$

Now, by the principle of (Art. 182), we must equate the coefficients of like powers of x, by which means we have

$$A_1a = \frac{m}{n}a^{\frac{m}{n}},$$

$$2A_2a + A_1 = \frac{m}{n}A_1,$$

$$3A_3a + 2A_2 = \frac{m}{n}A_2,$$

....

....

$$pA_pa + (p-1)A_{p-1} = \frac{m}{n}A_{p-1}.$$

$$\text{or,}\quad A_p = \frac{\left(\frac{m}{n} - p + 1\right)}{pa}A_{p-1}.$$

From this general value, we readily deduce the following:

$$\mathcal{A}_1 = \frac{m}{n} \cdot a^{\frac{m}{n}-1},$$

$$\mathcal{A}_2 = \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2} \cdot a^{\frac{m}{n}-2},$$

$$\mathcal{A}_3 = \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3} \cdot a^{\frac{m}{n}-3},$$

The general value being

$$\mathcal{A}_p = \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)\left(\frac{m}{n}-3\right)\ldots\left(\frac{m}{n}-p+1\right)}{2.3.4\ldots\ldots\ldots\ldots(p-1).p} \cdot a^{\frac{m}{n}-p}. \quad (15)$$

These values of $\mathcal{A}_1$; $\mathcal{A}_2$; $\mathcal{A}_3$; &c., substituted in (2), we have

$$(a+x)^{\frac{m}{n}} = a^{\frac{m}{n}} + \frac{m}{n} \cdot a^{\frac{m}{n}-1} x + \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2} \cdot a^{\frac{m}{n}-2} x^2 + \text{&c.} \quad \text{(A)}$$

If $n=1$, this value of (A) becomes

$$(a+x)^m = a^m + ma^{m-1}x + \frac{m(m-1)}{2} \cdot a^{m-2}x^2 + \text{&c.} \quad \text{(B)}$$

If $m=1$, then (A) becomes

$$(a+x)^{\frac{1}{n}} = a^{\frac{1}{n}} + \frac{1}{n} a^{\frac{1}{n}-1} x + \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)}{2} \cdot a^{\frac{1}{n}-} x^2 + \text{&c.} \quad \text{(C)}$$

The coefficient of the $(p+1)$th term as given by (15), becomes when $n=1$,

$$\frac{m(m-1)(m-2)(m-3)\ldots(m-p+2)(m-p+1)}{2.3.4\ldots\ldots\ldots\ldots(p-1).p}. \quad (16)$$

(187.) The numerator of this coefficient being formed of factors decreasing regularly by one, it follows that when $p=m+1$ it will vanish, and then the series must terminate; so that the number of terms of the expansion (B) will be $m+1$. But when $\frac{m}{n}$ is fractional, or a negative integer, the number of terms of the expansion must be infinite.

When a or x becomes negative, then those terms of the expansion will change signs, which contain odd powers of this negative quantity.

(188.) If in (B), we write a for x and x for a, we shall have

$$(x+a)^m=x^m+mx^{m-1}a+\frac{m(m-1)}{2}x^{m-2}a^2+\text{\&c.} \qquad (17)$$

Now, since the left-hand members of (B) and (17) are evidently equal, their right-hand members must be; and since, when m is a positive integer, the number of terms of (B) as well as (17) is equal to $m+1$, it follows that the terms of the expansion (B) must be *homogeneous* and symmetrical, and therefore of this form

$$(a+x)^m= \\ a^m+ma^{m-1}x+\frac{m(m-1)}{2}a^{m-2}x^2+\ldots\ldots+max^{m-1}+x^m. \qquad \text{(D)}$$

If in (A) we suppose $a=x=1$, we shall find

$$(1+1)^{\frac{m}{n}}=2^{\frac{m}{n}}=1+\frac{m}{n}+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2}+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}+\text{\&c} \qquad \text{(E)}$$

Therefore, in any expansion of a binomial, whose terms are both positive, the sum of the coefficients is equal to the same power, or root of 2.

(189.) If in (A), we suppose $a=1$; $x=-1$, we shall have

$$(1-1)^{\frac{m}{n}}=0=1-\frac{m}{n}+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2}-\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}+\&c. \quad \text{(F)}$$

That is, in any expansion of a binomial, one of whose terms is negative, the sum of the coefficients is $=0$; and therefore the sum of the positive coefficients must be equal to the sum of the negative ones.

(190.) By inspecting formula (A), we discover that the coefficients may be found in succession by the following

RULE.

Multiply any coefficient of any term by the exponent of the leading quantity in that term, and divide the product by the exponent of the following quantity diminished by one, and the result will be the coefficient of the succeeding term.

APPLICATION OF THE BINOMIAL THEOREM.

(191.) We will now make an application of this theorem; and, first, suppose in the expression (B), of page 246, we make successively $m=1$, 2, 3, and 4, and the results will be precisely the same as those first given on page 243. If we make in succession $m=5$, 6, and 7, we shall obtain the following results:

$$(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5,$$

$$(a+x)^6=$$
$$a^6+6a^5x+15a^4x^2+20a^3x^3+15a^2x^4+6ax^5+x^6,$$

$$(a+x)^7=$$
$$\left\{\begin{matrix}a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4 \\ +21a^2x^5+7ax^6+x^7.\end{matrix}\right\}$$

EXAMPLES.

1. Required the expansion of $(a+x)^{\frac{1}{2}}$.

In formula (C), make $n=2$, and it becomes

$$(a+x)^{\frac{1}{2}}=a^{\frac{1}{2}}+\frac{1}{2}a^{-\frac{1}{2}}x-\frac{1}{2.4}\cdot a^{-\frac{3}{2}}x^2+\frac{3}{2.4.6}\cdot a^{-\frac{5}{2}}x^3-\&c.$$

$$=a^{\frac{1}{2}}\left\{1+\frac{1}{2}a^{-1}x-\frac{1}{2.4}\cdot a^{-2}x^2+\frac{3}{2.4.6}\cdot a^{-3}x^3-\&c.\right\}$$

Writing the different powers of a, which have negative exponents, in the denominator, by which means their exponents change signs and become positive (Art. 49), and we find

$$(a+x)^{\frac{1}{2}}=a^{\frac{1}{2}}\left\{1+\frac{x}{2a}-\frac{x^2}{2.4a^2}+\frac{3x^3}{2.4.6a^3}-\frac{3.5x^4}{2.4.6.8a^4}+\&c.\right\}$$

2. Required to expand $(a+x)^{\frac{1}{3}}$.

Changing n into 3, in (C), and we have

$$(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}+\frac{1}{3}a^{-\frac{2}{3}}x-\frac{2}{3.6}a^{-\frac{5}{3}}x^2+\frac{2.5}{3.6.9}\cdot a^{-\frac{8}{3}}x^3-\&c.$$

Removing the factor $a^{\frac{1}{3}}$, and causing the different powers of a to pass into the denominators, as in the last example, we obtain

$$(a+x)^{\frac{1}{3}}=a^{\frac{1}{3}}\left\{1+\frac{x}{3a}-\frac{2x^2}{3.6a^2}+\frac{2.5x^3}{3.6.9a^3}-\frac{2.5.8x^4}{3.6.9.12a^4}+\&c.\right\}$$

3. Expand $(a+x)^{\frac{1}{4}}$.

Making $n=4$ in (C), and it becomes

$$(a+x)^{\frac{1}{4}}=a^{\frac{1}{4}}+\frac{1}{4}\cdot a^{-\frac{3}{4}}x-\frac{3}{4.8}a^{-\frac{7}{4}}x^2+\frac{3.7}{4.8.12}a^{-\frac{11}{4}}x^3-\&c.$$

Or,

$$(a+x)^{\frac{1}{4}}=a^{\frac{1}{4}}\left\{1+\frac{x}{4a}-\frac{3x^2}{4.8a^2}+\frac{3.7x^3}{4.8.12a^3}-\&c.\right\}$$

4. Required the expansion of $(1+x)^{\frac{1}{2}}$.

This example will agree with example 1, if we write 1 for a. Making this change in example 1, we get

$$(1+x)^{\frac{1}{2}}=1+\frac{x}{2}-\frac{x^2}{2.4}+\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}+\&c.$$

5. Required the expansion of $(1+1)^{\frac{1}{2}}$ or $\sqrt{2}$.

In the last example make $x=1$, and it becomes

$$(1+1)^{\frac{1}{2}}=\sqrt{2}=1+\frac{1}{2}-\frac{1}{2.4}+\frac{3}{2.4.6}-\frac{3.5}{2.4.6.8}+\&c.$$

6. Required the expansion of $\sqrt{1-x}$ or $(1-x)^{\frac{1}{2}}$.

In example 4, change x into $-x$, and we get

$$(1-x)^{\frac{1}{2}}=1-\frac{x}{2}-\frac{x^2}{2.4}-\frac{3x^3}{2.4.6}-\frac{3.5x^4}{2.4.6.8}-\&c.$$

This expansion agrees with the one found by indeterminate coefficients. (See Ex. 3, Page 240.)

7. Expand $(a+x)^{-4}$.

In (B), make $m=-4$, and it becomes

$$(a+x)^{-4}$$
$$=a^{-4}-4a^{-5}x+10a^{-6}x^2-20a^{-7}x^3+35a^{-8}x^4-\&c.$$

$$=\frac{1}{a^4}\left\{1-\frac{4x}{a}+\frac{10x^2}{a^2}-\frac{20x^3}{a^3}+\frac{35x^4}{a^4}-\&c.\right\}$$

8. Required to expand $\dfrac{1}{a+x}$ or $(a+x)^{-1}$.

Making $m=-1$ in (B), we find

$$(a+x)^{-1}$$
$$=a^{-1}-a^{-2}x+a^{-3}x^2-a^{-4}x^3+a^{-5}x^4-a^{-6}x^5+\text{\&c.}$$
$$=\frac{1}{a}\left\{1-\frac{x}{a}+\frac{x^2}{a_2}-\frac{x^3}{a^3}+\frac{x^4}{a^4}-\frac{x^5}{a^5}+\text{\&c.}\right\}$$

9. Required the expansion of $\frac{1}{1-x}$.

In the last example, write 1 for a, and $-x$ for x, and it becomes

$$\frac{1}{1-x}=1+x+x^2+x^3+x^4+x^5+x^6+\text{\&c.}$$

10. What is the expansion of $(a-b)^{\frac{1}{4}}$?

Ans. $a^{\frac{1}{4}}\left\{1-\frac{b}{4a}-\frac{3b^2}{4.8a^2}-\frac{3.7b^3}{4.8.12a^3}-\frac{3.7.11b^4}{4.8.12.16a^4}-\text{\&c.}\right\}$

11. What is the expansion of $(a+x)^{-\frac{1}{5}}$?

$\frac{1}{a^{\frac{1}{5}}}\left\{1-\frac{x}{5a}+\frac{6x^2}{5.10a^2}-\frac{6.11x^3}{5.10.15a^3}+\frac{6.11.16x^4}{5.10.15.20a^4}-\text{\&c.}\right\}$

12. What is the expansion of $(a^3-x)^{\frac{1}{2}}$?

Ans. $a^{\frac{3}{2}}\left\{1-\frac{x}{2a^3}-\frac{x^2}{2.4a^6}-\frac{3x^3}{2.4.6a^9}-\frac{3.5x^4}{2.4.6.8a^{12}}-\text{\&c.}\right\}$

13. What is the expansion of $(p+q\sqrt{-1})^{\frac{1}{3}}$?

If in example 2, we change a into p, and x into $q\sqrt{-1}$, we shall find, by reccollecting that by Art. 126, we have

$$(q\sqrt{-1})^2=-q^2;\ (q\sqrt{-1})^3=-q^3\sqrt{-1};$$
$$(q\sqrt{-1})^4=q^4;\ (q\sqrt{-1})^5=q^5\sqrt{-1};\ \text{\&c.}$$

$$(p+q\sqrt{-1})^{\frac{1}{3}}$$

$$=p^{\frac{1}{3}}+\frac{1}{3}p^{-\frac{2}{3}}q\sqrt{-1}+\frac{2}{3.6}p^{-\frac{5}{3}}q^2-\frac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3\sqrt{-1}-\&c$$

14. What is the expansion of $(p-q\sqrt{-1})^{\frac{1}{3}}$?

Changing, in example 2, a into p, and x into $-q\sqrt{-1}$, we easily find

$$(p-q\sqrt{-1})^{\frac{1}{3}}$$

$$=p^{\frac{1}{3}}-\frac{1}{3}p^{-\frac{2}{3}}q\sqrt{-1}+\frac{2}{3.6}p^{-\frac{5}{3}}q^2+\frac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3\sqrt{-1}-\&c.$$

(192.) This theorem may be applied to quantities of more than two terms.

Suppose we wish the expansion of $(a+b+c)^3$.

Assume

$$d=b+c,$$

and

$$(a+b+c)^3=(a+d)^3.$$

Now, in (B), make $x=d$, and $m=3$, and it will become

$$(a+d)^3=a^3+3a^2d+3ad^2+d^3. \qquad (1)$$

Now, by assumption $d=b+c$; therefore we have

$$d^2=b^2+2bc+c^2,$$

and

$$d^3=b^3+3b^2c+3bc^2+c^3.$$

These values of d, d^2 and d^3, being substituted in (1), we get

$$(a+b+c)^3=$$
$$a^3+3a^2(b+c)+3a(b^2+2bc+c^2)+b^3+3b^2c+3bc^2+c^3$$
$$=\left\{\begin{array}{l}a^3+3a^2b+3a^2c+3ab^2+6abc+3ac^2\\ \qquad\qquad\qquad +b^3+3b^2c+3bc^2+c^3.\end{array}\right\}$$

We might proceed in this way to obtain the expansion of algebraic polynomials of any number of terms, but a better method will be to deduce a multinomial theorem, which may be done as follows:

MULTINOMIAL THEOREM.

(193.) This theorem, as we have just hinted, gives the law of the expansion of

$$(a_0 + a_1x + a_2x^2 + a_3x^3 + \&c.)^{\frac{m}{n}},$$

or of any other polynomial, having for an exponent any value whatever. To determine this law, assume

$$(a_0 + a_1x + a_2x^2 + \&c.)^{\frac{m}{n}} = A_0 + A_1x + A_2x^2 + \&c. \qquad (1)$$

When $x = 0$, we have $a_0^{\frac{m}{n}} = A_0$; therefore we have

$$(a_0 + a_1x + a_2x^2 + \&c.)^{\frac{m}{n}} = a_0^{\frac{m}{n}} + A_1x + A_2x^2 + \&c. \qquad (2)$$

Writing x_1 for x, we have

$$(a_0 + a_1x_1 + a_2x_1^2 + \&c.)^{\frac{m}{n}} = a_0^{\frac{m}{n}} + A_1x_1 + A_2x_1^2 + \&c. \qquad (3)$$

Subtracting (3) from (2), we find

$$\left.\begin{aligned}(a_0 + a_1x + a_2x^2 + \&c.)^{\frac{m}{n}} - (a_0 + a_1x_1 + a_2x_1^2 + \&c.)^{\frac{m}{n}} \\ = A_1(x - x_1) + A_2(x^2 - x_1^2) + \&c.\end{aligned}\right\} \qquad (4)$$

If we suppose

$$U = (a_0 + a_1x + a_2x^2 + \&c.)^{\frac{1}{n}},$$

$$U_1 = (a_0 + a_1x_1 + a_2x_1^2 + \&c.)^{\frac{1}{n}},$$

we readily find

$$U^m - U_1^m$$
$$= (a_0 + a_1 x + a_2 x^2 + \&c.)^{\frac{m}{n}} - (a_0 + a_1 x_1 + a_2 x_1^2 + \&c.)^{\frac{m}{n}}.$$
$$U^n - U_1^n = a_1(x - x_1) + a_2(x^2 - x_1^2) + \&c.$$

Hence (4) becomes

$$\left.\begin{array}{l} U^m - U_1^m = \\ A_1(x - x_1) + A_2(x^2 - x_1^2) + A_3(x^3 - x_1^3) + \&c. \end{array}\right\} \qquad (5)$$

Dividing the left-hand member of (5) by $U^n - U_1^n$, and its right-hand member by its equal

$$a_1(x - x_1) + a_2(x^2 - x_1^2) + \&c.,$$

we get

$$\left.\begin{array}{l} \dfrac{U^m - U_1^m}{U^n - U_1^n} = \\ \dfrac{A_1(x - x_1) + A_2(x^2 - x_1^2) + A_3(x^3 - x_1^3) + \&c.}{a_1(x - x_1) + a_2(x^2 - x_1^2) + a_3(x^3 - x_1^3) + \&c.} \end{array}\right\} \qquad (6)$$

If we divide both numerator and denominator of the left-hand member of (6) by $U - U_1$, it will become [see formula (B), Art. 185],

$$\frac{U^{m-1} + U_1 U^{m-2} + \ldots . U_1^{m-1}}{U^{n-1} + U_1 U^{n-2} + \ldots . U_1^{n-1}}. \qquad (7)$$

If we divide both numerator and denominator of the right-hand member of (6) by $x - x$, it will become

$$\frac{A_1 + A_2(x + x_1) + A_3(x^2 + x x_1 + x_1^2) + \&c.}{a_1 + a_2(x + x_1) + a_3(x^2 + x x_1 + x_1^2) + \&c.} \qquad (8)$$

The expressions (7) and (8) are equal. Now, when $x = x_1$, the expression (7) becomes

$$\frac{m U^{m-1}}{n U^{n-1}} = \frac{m}{n} \cdot \frac{U^m}{U^n},$$

which, by re-substituting the value of U, becomes

$$\frac{m}{n}\cdot\frac{(a_0+a_1x+a_2x^2+\text{ \&c.})^{\frac{m}{n}}}{a_0+a_1x+a_2x^2+\text{ \&c.}}. \tag{9}$$

When $x=x_1$, the expression (8) becomes

$$\frac{A_1+2A_2x+3A_3x^2+\text{ \&c.}}{a_1+2a_2x+3a_3x^2+\text{ \&c.}}. \tag{10}$$

Equating the expressions (9) and (10), and clearing of fractions, we have

$$\left.\begin{array}{l}\frac{m}{n}(a_0+a_1x+a_2x^2+\text{\&c})^{\frac{m}{n}}\,.\,(a_1+2a_2x+3a_3x^2+\text{\&c})= \\ (a_0+a_1x+a_2x^2+\text{\&c})\,.\,(A_1+2A_2x+3A_3x^2+\text{\&c.})\end{array}\right\} \tag{11}$$

Multiplying (1) by $\frac{m}{n}$, we have

$$\frac{m}{n}(a_0+a_1x+a_2x^2+\text{\&c.})^{\frac{m}{n}}=\frac{m}{n}(A_0+A_1x+A_2x^2+\text{\&c.}) \tag{12}$$

Hence (11) becomes

$$\left.\begin{array}{l}\frac{m}{n}(A_0+A_1x+A_2x^2+\text{\&c}).(a_1+2a_2x+3a_3x^2+\text{\&c})= \\ (a_0+a_1x+a_2x^2+\text{\&c}).(A_1+2A_2x+3A_3x^2+\text{\&c.})\end{array}\right\} \tag{13}$$

By actual multiplication (13) becomes

$$\left.\begin{array}{l}\frac{m}{n}A_0a_1+\left.\begin{array}{r}A_1a_1\\2A_0a_2\\ \\ \end{array}\right|\frac{m}{n}x+\left.\begin{array}{r}A_2a_1\\2A_1a_2\\3A_0a_3\\ \end{array}\right|\frac{m}{n}x^2+\left.\begin{array}{r}A_3a_1\\2A_2a_2\\3A_1a_3\\4A_0a_4\end{array}\right|\frac{m}{n}x^3+\text{ \&c.} \\ \\ =A_1a_0+\left.\begin{array}{r}2A_2a_0\\A_1a_1\\ \\ \end{array}\right|x+\left.\begin{array}{r}3A_3a_0\\2A_2a_1\\A_1a_2\\ \end{array}\right|x^2+\left.\begin{array}{r}4A_4a_0\\3A_3a_1\\2A_2a_2\\A_1a_3\end{array}\right|x^3+\text{ \&c.}\end{array}\right\} \tag{14}$$

Equating coefficients of like powers of x in (14), we have

$$\mathcal{A}_1 a_0 = \frac{m}{n}\mathcal{A}_0 a_1.$$

$$2\mathcal{A}_2 a_0 + \mathcal{A}_1 a_1 = \frac{m}{n}\mathcal{A}_1 a_1 + 2\frac{m}{n}\mathcal{A}_0 a_2.$$

$$\left.\begin{array}{l} 3\mathcal{A}_3 a_0 + 2\mathcal{A}_2 a_1 + \mathcal{A}_1 a_2 = \\ \qquad \frac{m}{n}\mathcal{A}_2 a_1 + 2\frac{m}{n}\mathcal{A}_1 a_2 + 3\frac{m}{n}\mathcal{A}_0 a_3. \end{array}\right\}$$

$$\left.\begin{array}{l} 4\mathcal{A}_4 a_0 + 3\mathcal{A}_3 a_1 + 2\mathcal{A}_2 a_2 + \mathcal{A}_1 a_3 = \\ \qquad \frac{m}{n}\mathcal{A}_3 a_1 + 2\frac{m}{n}\mathcal{A}_2 a_2 + 3\frac{m}{n}\mathcal{A}_1 a_3 + 4\frac{m}{n}\mathcal{A}_0 a_4. \end{array}\right\}$$

....

....

If for $\mathcal{A}_0$, we use its equal $a_0^{\frac{m}{n}}$, we shall find from the above system of equations

$$\mathcal{A}_0 = a_0^{\frac{m}{n}},$$

$$\mathcal{A}_1 = \frac{m}{n} a_0^{\frac{m}{n}-1} a_1,$$

$$\mathcal{A}_2 = \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2} a_0^{\frac{m}{n}-2} a_1^2 + \frac{m}{n} a_0^{\frac{m}{n}-1} a_2,$$

$$\mathcal{A}_3 = \left\{\begin{array}{l} \frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3} a_0^{\frac{m}{n}-3} a_1^3, \\ + \frac{m}{n}\left(\frac{m}{n}-1\right) a_0^{\frac{m}{n}-2} a_1 a_2 + \frac{m}{n} a_0^{\frac{m}{n}-1} a_3. \end{array}\right.$$

....

These values of A_0, A_1, A_2, A_3, &c., substituted in (1), we have

$$\left.\begin{array}{l}(a_0+a_1x+a_2x^2+a_3x^3+\&c.)^{\frac{m}{n}}=a_0^{\frac{m}{n}}+\frac{m}{n}a_0^{\frac{m}{n}-1}a_1x+ \\ \left\{\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{2}a_0^{\frac{m}{n}-2}a_1^2+\frac{m}{n}a_0^{\frac{m}{n}-1}a_2\right\}x^2+ \\ \left\{\begin{array}{l}\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{2.3}a_0^{\frac{m}{n}-3}a_1^3 \\ +\frac{m}{n}\left(\frac{m}{n}-1\right)a_0^{\frac{m}{n}-2}a_1a_2+\frac{m}{n}a_0^{\frac{m}{n}-1}a_3\end{array}\right\}x^3+\&c.\end{array}\right\}\text{(A)}$$

EXAMPLES.

1. What is the cube of $1+x+x^2+x^3+x^4+\&c.$?

If, in our general expression (A) of the multinomial theorem, we make $a_0=a_1=a_2=a_3=\&c.=1$; and $m=3, n=1$, we shall have

$$(1+x+x^2+x^3+x^4+\&c.)^3=1+3x+6x^2+10x^3+\&c.$$

2. What is the square root of $1+x+x^2+x^3+\&c.$?

In our general expression (A), we must have $m=1, n=2$, and $1=a_0=a_1=a_2=a_3=\&c.$

$$(1+x+x^2+x^3+\&c.)^{\frac{1}{2}}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\&c.$$

3. What is the cube root of $1+x+x^2+x^3+\&c.$?

In (A), make $m=1, n=3$, and $1=a_0=a_1=a_2=a_3=\&c.$, and we get

$$(1+x+x^2+x^3+\&c.)^{\frac{1}{3}}=1+\frac{1}{3}x+\frac{2}{9}x^2+\frac{14}{81}x^3+\&c.$$

4. What is the cube root of $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\&c.$

Ans. $1+\frac{1}{6}x+\frac{1}{12}x^2+\frac{35}{648}x^3+\&c.$

REVERSION OF SERIES.

(194.) Suppose we have

$$a_1x+a_2x^2+a_3x^3+a_4x^4+\&c.=y. \qquad (1)$$

The process by which we find x in terms of y, is called reverting the series (1), which may be effected by the following method :

Assume

$$x=A_1y+A_2y^2+A_3y^3+A_4y^4+\&c. \qquad (2)$$

Now, we find by actual multiplication, or by means of the multinomial theorem,

$$x^2=A_1^2y^2+2A_1A_2y^3+\left.\begin{matrix}2A_1A_3\\+A_2^2\end{matrix}\right\}y^4+\&c.$$

$$x^3=A_1^3y^3+3A_1^2A_2y^4+\&c.$$
$$x^4=A_1^4y^4+\&c.$$

These values of x, x^2, x^3, x^4, &c., substituted in (1), we have

$$\left.\begin{array}{l|l|l|l}
A_1a_1y+A_2a_1 & y^2+\quad A_3a_1 & y^3+\quad A_4a_1 & y^4+\&c. \\
\qquad\quad A_1^2a_2 & +2A\ A_2a_2 & +\ 2A_1A_3a_2 & \\
 & \quad +A_1^3a_3 & \quad +A_2^2a_2 & \\
 & & +3A_1^2A_2a_3 & \\
 & & \quad +A_1^4a_4 &
\end{array}\right\}=y. \qquad (3)$$

Hence, we have by the method of indeterminate coefficients, (Art. 182.)

$$\mathcal{A}_1 a_1 = 1,$$
$$\mathcal{A}_2 a_1 + \mathcal{A}_1^2 a_2 = 0,$$
$$\mathcal{A}_3 a_1 + 2\mathcal{A}_1\mathcal{A}_2 a_2 + \mathcal{A}_1^3 a_3 = 0,$$
$$\mathcal{A}_4 a_1 + 2\mathcal{A}_1\mathcal{A}_3 a_2 + \mathcal{A}_2^2 a_2 + 3\mathcal{A}_1^2\mathcal{A}_2 a_3 + \mathcal{A}_1^4 a_4 = 0.$$
....

From the above conditions we deduce

$$\mathcal{A}_1 = \frac{1}{a_1},$$

$$\mathcal{A}_2 = -\frac{a_2}{a_1^3},$$

$$\mathcal{A}_3 = \frac{2a_2^2 - a_1 a_3}{a_1^5},$$

$$\mathcal{A}_4 = -\frac{5a_2^3 - 5a_1 a_2 a_3 + a_1^2 a_4}{a_1^7}.$$
....

These values of $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, &c., substituted in (2), we have

$$x = \left\{ \begin{array}{l} \frac{1}{a_1}y - \frac{a_2}{a_1^3}y^2 + \frac{2a_2^2 - a_1 a_3}{a_1^5}y^3 \\ - \frac{5a_2^3 - 5a_1 a_2 a_3 + a_1^2 a_4}{a_1^7}y^4 + \text{\&c.} \end{array} \right\} \qquad (A)$$

So that if (1) is true for all values of x and y, then also will (A) be true for all values of x and y; and such is the general relation between two series when one is the reversion of the other.

EXAMPLES.

1. Given the series $x + x^2 + x^3 + x^4 + \text{\&c.} = y$, to find its reversion, that is, to find the value of x in terms of y.

Comparing this series with the series (1) of this article, we see that

$$1=a_1=a_2=a_3=a_4=\&c.,$$

these values substituted in (A), give

$$x=y-y^2+y^3-y^4+\ \&c.$$

2. Given $x-\frac{1}{2}x^2+\frac{1}{4}x^3-\frac{1}{8}x^4+\ \&c.=y$ to find x.

In this example we have

$$a_1=1\ ;\ a_2=-\tfrac{1}{2}\ ;\ a_3=\tfrac{1}{4}\ ;\ a_4=-\tfrac{1}{8}\ ;\ \&c.,$$

which values substituted in (A), give

$$x=y+\tfrac{1}{2}y^2+\tfrac{1}{4}y^3+\tfrac{1}{8}y^4+\ \&c.$$

3. Given $1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\&c.=y$, to find x in terms of y.

In this example we first transpose the 1, by which means we have

$$x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\&c.=y-1=y'.$$

This, compared with (1), we find

$$a_1=1\ ;\ a_2=\frac{1}{2}\ ;\ a_3=\frac{1}{2.3}\ ;\ a_4=\frac{1}{2.3.4}\ ;\ \&c.$$

These values cause (A) to become

$$x=y'-\frac{y'^2}{2}+\frac{y'^3}{3}-\frac{y'^4}{4}\&c.;$$

or restoring the value of y',

$$x=(y-1)-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}-\frac{(y-1)^4}{4}+\&c.$$

4. Given $x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\&c.=y$, to find x.

Ans. $x=y-\frac{y^2}{2}+\frac{y^3}{2.3}-\frac{y^4}{2.3.4}+\&c.$

DIFFERENTIAL METHOD.

(195.) This method shows how to find any particular term of a regular increasing series, or the sum of a certain number of terms.

If we take the regular increasing series

$$a_1\,;\ a_2\,;\ a_3\,;\ a_4\,;\ a_5\,;\ \&c., \qquad (1)$$

and subtract each term from the next succeeding one, we shall obtain the following series, which we shall call the *first order* of differences:

$$a_2-a_1\,;\ a_3-a_2\,;\ a_4-a_3\,;\ a_5-a_4\,;\ \&c. \qquad (2)$$

Again, subtracting each term of this series from the next succeeding term, and we find for the *second order* of differences

$$a_3-2a_2+a_1\,;\ a_4-2a_3+a_2\,;\ a_5-2a_4+a_3\,;\ \&c. \qquad (3)$$

Subtracting again each term of series (3), of the second order of differences, from its next succeeding term, and we get a series of *third order* of differences, as follows:

$$a_4-3a_3+3a_2-a_1\,;\ a_5-3a_4+3a_3-a_2\,;\ \&c. \qquad (4)$$

Subtracting once more we find, for the *fourth order* of differences,

$$a_5-4a_4+6a_3-4a_2+a_1\,;\ \&c. \qquad (5)$$

If we take only the first terms of the series (2), (3), (4), (5), and represent them respectively by D_1; D_2; D_3; D_4; &c., we shall have

$$\left.\begin{aligned}
D_1 &= a_2-a_1,\\
D_2 &= a_3-2a_2+a_1,\\
D_3 &= a_4-3a_3+3a_2-a_1,\\
D_4 &= a_5-4a_4+6a_3-4a_2+a_1,\\
&\&c. \quad \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \qquad (6)$$

The coefficients of the different terms which constitute the right-hand members of equations (6) are the same as the coefficients of the different terms of the expansion of the binomial $(1-1)^n$, whose expanded form is

$$1-n+\frac{n(n-1)}{2}-\frac{n(n-1)(n-2)}{2.3}+\frac{n(n-1)(n-2)(n-3)}{2.3.4}-\&c.$$

Hence, the general equation of (6) is

$$D_n=\left\{\begin{array}{c} a_{n+1}-na_n+\frac{n(n-1)}{2}a_{n-1}-\frac{n(n-1)(n-2)}{2.3}a_{n-2} \\ +\frac{n(n-1)(n-2)(n-3)}{2.3.4}a_{n-3}-\&c. \end{array}\right\} \quad (7)$$

If the terms of the right-hand members of (6) are taken in a reverse order, we shall have

When n is an even number,

$$D_n=\left\{\begin{array}{c} a_1-na_2+\frac{n(n-1)}{2}a_3-\frac{n(n-1)(n-2)}{2.3}a_4 \\ +\frac{n(n-1)(n-2)(n-3)}{2.3.4}a_5-\&c. \end{array}\right\} \quad (A)$$

When n is an odd number,

$$D_n=\left\{\begin{array}{c} -a_1+na_2-\frac{n(n-1)}{2}a_3+\frac{n(n-1)(n-2)}{2.3}a_4 \\ -\frac{n(n-1)(n-2)(n-3)}{2.3.4}a_5+\&c. \end{array}\right\} \quad (B)$$

EXAMPLES.

1. Required the first term of the fourth order of differences of the series 1, 8, 27, 64, 125, &c.

In this example we have

$a_1=1$; $a_2=8$; $a_3=27$; $a_4=64$; $a_5=125$ and $n=4$

These values substituted in the formula (A), since n is even, give

$$D_4 = 1 - 4.8 + \frac{4.3}{2}\,.\,27 - \frac{4.3.2}{2.3}\,.\,64 + \frac{4.3.2.1}{2.3.4}\,.\,125 = 0.$$

2. Required the first term of the third order of differences of the series $1, 2^4, 3^4, 4^4$, &c.

Ans. 60.

3. Required the first term of the fourth order of differences of the series 1, 6, 20, 50, 105, &c.

Ans. 2.

(196.) To find the nth term of the series

$$a_1\,;\ a_2\,;\ a_3\,;\ a_4\,;\ a_5\,;\ \&c.,$$

we proceed as follows:

From the first of the equations (6) of last article, we obtain

$$a_2 = a_1 + D_1\,;$$

this value of a_2 substituted in the second of equations (6), gives

$$a_3 = a_1 + 2D_1 + D_2\,;$$

proceeding in this way we have the following:

$$\left.\begin{aligned}
a_1 &= a_1,\\
a_2 &= a_1 + D_1,\\
a_3 &= a_1 + 2D_1 + D_2,\\
a_4 &= a_1 + 3D_1 + 3D_2 + D_3,\\
a_5 &= a_1 + 4D_1 + 6D_2 + 4D_3 + D_4,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\}\quad(8)$$

Where the coefficients of the terms of the value of a_n are equal to the coefficients of the terms of the expansion of the binomial $(1+1)^{n-1}$, whose expanded form is

$$\left.\begin{array}{l}1+(n-1)+\dfrac{(n-1)(n-2)}{2}+\dfrac{(n-1)(n-2)(n-3)}{2.3} \\ \qquad +\dfrac{(n-1)(n-2)(n-3)(n-4)}{2.3.4}+\&c.\end{array}\right\} \quad (9)$$

Therefore, we have

$$a_n=\left\{\begin{array}{l}a_1+(n-1)D_1+\dfrac{(n-1)(n-2)}{2}D_2 \\ +\dfrac{(n-1)(n-2)(n-3)}{2.3}D_3+\&c.\end{array}\right\} \quad (C)$$

EXAMPLES.

1. Required the tenth term of the series

1, 4, 8, 13, 19, &c.

$a_1 = 1,\ 4,\ 8,\ 13,\ 19,$
$D_1 = 3,\ 4,\ 5,\ 6,$
$D_2 = 1,\ 1,\ 1,$
$D_3 = 0,\ 0.$

Hence, in this example,

$a_1 = 1$; $D_1 = 3$; $D_2 = 1$; $D_3 = 0$; and $n = 10$,

which values being substituted in (C), we find

$a_{10} = 1+9.3+\dfrac{9.8}{2} = 64$, for the tenth term required.

2. Required the nth term of the series 2, 6, 12, 20, 30, &c.

$a_1 = 2,\ 6,\ 12,\ 20,$
$D_1 = 4,\ 6,\ 8,$
$D_2 = 2,\ 2,$
$D_3 = 0.$

These values substituted in (C), give

$a_n = 2+(n-1).4+\dfrac{(n-1)(n-2)}{2}.2 = n^2+n = n(n+1),$

which is the nth term sought.

3. What is the nth term of the series

$$1,\ 3,\ 6,\ 10,\ 15,\ 21,\ \&c.?$$

Ans. $\dfrac{n(n+1)}{2}$.

(197.) To find the sum of n terms of the series

$$a_1\ ;\ a_2\ ;\ a_3\ ;\ a_4\ ;\ a_5\ ;\ \&c.$$

we operate as follows:

Take the new series

$$0\ ;\ a_1\ ;\ a_1+a_2\ ;\ a_1+a_2+a_3\ ;\ a_1+a_2+a_3+a_4\ \&c. \qquad (10)$$

Subtracting each term from its next succeeding term, we have

$$a_1\ ;\ a_2\ ;\ a_3\ ;\ a_4\ ;\ a_5\ ;\ \&c.$$

which is the same as the original series; hence, the $n+1$ difference of the series (10), is the same as the n difference of the proposed series; therefore, if in the formula (C), we change a_1 into 0, n into $n+1$, D_1 into a_1, D_2 into D_1, D_3 into D_2, &c., we shall have

$$A_{n+1}=\left\{\begin{array}{l} na_1+\dfrac{n(n-1)}{2}D_1+\dfrac{n(n-1)(n-2)}{2.3}D_2 \\ \quad +\dfrac{n(n-1)(n-2)(n-3)}{2.3.4.}D_3+\ \&c. \end{array}\right\} \qquad (11)$$

which expresses the $n+1$th term of the series (10), but the $n+1$th term of the series (10) is the same as the sum of n terms of the series

$$a_1\ ;\ a_2\ ;\ a_3\ ;\ a_4\ ;\ a_5\ ;\ \&c.$$

Putting this sum equal to S_n, we shall have for the sum of n terms of the above series, the following expression:

$$S_n=\left\{\begin{array}{l} na_1+\dfrac{n(n-1)}{2}D_1+\dfrac{n(n-1)(n-2)}{2.3}D_2 \\ \quad +\dfrac{n(n-1)(n-2)(n-3)}{2.3.4}D_3+\ \&c. \end{array}\right\} \qquad (D)$$

EXAMPLES.

1. Required the sum of n terms of the series

$$1,\ 3,\ 5,\ 7,\ 9,\ \&c.$$

$$a_1 = 1,\ 3,\ 5,$$
$$D_1 = 2,\ 2,$$
$$D_2 = 0.$$

These values substituted in (D), give

$S_n = n + n(n-1) = n^2$, for the sum of n terms sought.

2. Required the sum of n terms of the series

$$1,\ 3,\ 6,\ 10,\ 15,\ \&c.$$

$$a_1 = 1,\ 3,\ 6,\ 10,$$
$$D_1 = 2,\ 3,\ 4,$$
$$D_2 = 1,\ 1,$$
$$D_3 = 0.$$

These values substituted in (D), give

$$S_n = n + n(n-1) + \frac{n(n-1)(n-2)}{2.3} = \frac{n(n+1)(n+2}{2.3}.$$

3. What is the sum of n terms of the series

$$1,\ 2^4,\ 3^4,\ 4^4,\ \&c.?$$

Ans. $\frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}$.

4. What is the sum of n terms of the series

$$1,\ 2,\ 3,\ 4,\ 5,\ \&c.?$$

Ans. $\frac{n(n+1)}{2}$.

5. What is the sum of n terms of the series

$$1,\ 2^3,\ 3^3,\ 4^3,\ \&c.?$$

Ans. $\left\{\frac{n(n+1)}{2}\right\}^2$.

The answer of the fifth example, being the square of the answer to the fourth example, it follows that

$$\{1+2+3+4+5+\ldots n\}^2=1^3+2^3+3^3+4^3+5^3+\ldots n^3.$$

SUMMATION OF INFINITE SERIES.

(198.) *An Infinite Series* is a progression of quantities continued to an infinite number of terms, usually according to some regular law.

If each term of an infinite series is greater than its preceding term, the series is *diverging*.

In general, when each term is less than its preceding, the series is *converging*, but this is not always the case; for instance, the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+$ &c., which is called a *harmonic series*, is not a converging series, notwithstanding each term is less than its preceding one; its sum to infinity is itself infinite.

A neutral series is one whose terms are all equal, and their signs alternately $+$ and $-$, thus,

$$\frac{a}{2}=\frac{a}{1+1}=a-a+a-a+a-a+\text{\&c.}$$

An ascending series is one in which the powers of the unknown quantities ascend, as

$$a+bx+cx^2+dx^3+\text{\&c}$$

A descending series is one in which the powers of the unknown quantity descend, as

$$a+bx^{-1}+cx^{-2}+dx^{-3}+\text{\&c.},$$

or $$a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}+\text{\&c.}$$

(199.) If we take the difference between the two fractions $\frac{q}{r}$, $\frac{q}{r+p}$ we shall find $\frac{q}{r}-\frac{q}{r+p}=\frac{pq}{r(r+p)}$; hence

$$\frac{q}{r(r+p)}=\frac{1}{p}\left(\frac{q}{r}-\frac{q}{r+p}\right);$$

so that any fraction of the form $\frac{q}{r(r+p)}$ is equal to $\frac{1}{p}$th the difference between the two fractions $\frac{q}{r}$ and $\frac{q}{r+p}$; hence, it follows that if there be any series of fractions of the form $\frac{q}{r(r+p)}$, the sum of the series will equal $\frac{1}{p}$th the difference of a series of the form $\frac{q}{r}$ and another of the form $\frac{q}{r+p}$; so that whenever this difference can be found, the sum of the proposed series can be obtained.

EXAMPLES.

1. Required the sum of n terms of the series

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\text{\&c.}$$

In this example $q=1$; $p=1$; and r takes successively the values 1, 2, 3, 4, &c.

Hence, we have

$$\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n}\\ -\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n}+\frac{1}{n+1}\right)\end{array}\right\}=1-\frac{1}{n+1}=\frac{n}{n+1}$$

If $n=\infty$, then the sum $\frac{n}{n+1}$ becomes $=1$.

2. Find the sum of n terms of the series

$$\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\frac{1}{4.7}+\&c.$$

In this example $q=1$; $p=3$; and $r=1, 2, 3, 4$, &c.

$$\frac{1}{3}\left\{\begin{array}{l}1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots.\frac{1}{n} \\ \qquad -\left(\frac{1}{4}+\frac{1}{5}+\ldots.\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)\end{array}\right\}$$

$$=\frac{1}{3}\left(\frac{11}{6}-\frac{1}{n+1}-\frac{1}{n+2}-\frac{1}{n+3}\right),$$

which becomes, when n is infinite $\frac{11}{18}$.

3. Find the sum of n terms of $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\&c.$

Here we find $q=1$; $p=2$.

$$\frac{1}{2}\left\{\begin{array}{l}1+\frac{1}{3}+\frac{1}{5}+\ldots.\frac{1}{2n-1} \\ -\left(\frac{1}{3}+\frac{1}{5}+\ldots.\frac{1}{2n-1}+\frac{1}{2n+1}\right)\end{array}\right\}=\frac{1}{2}\left(1-\frac{1}{2n+1}\right).$$

Therefore,

$S_n=\frac{1}{2}\left(1-\frac{1}{2n+1}\right)$; and $S_\infty=\frac{1}{2}$ where S_n represents the sum of n terms, and S_∞ represents the sum of an infinite number of terms.

4. Required the sum of the series

$$1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\&c.$$

This series divided by 2, becomes

$$\frac{1}{2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\text{\&c.},$$

and the sum of this has already been found. (See example 1, of this Art.) Therefore, the sum of the proposed series is

$$S_n=2\left(1-\frac{1}{n+1}\right);\ S_\infty=2.$$

5. Find the sum of the series of

$$\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\text{\&c.}$$

$$\left\{\begin{array}{l}\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\ldots\ldots\mp\frac{n+1}{2n+1}\\ -\left(\frac{2}{5}-\frac{3}{7}+\frac{4}{9}-\ldots\ldots\pm\frac{n+1}{2n+1}\mp\frac{n+1}{2n+3}\right);\end{array}\right\}$$

this becomes

$$\frac{2}{3}\pm\frac{n+1}{2n+1}-(1-1+1-\ldots\ldots\pm 1.)$$

If we use the upper sign, the quantity within the parenthesis will $=1$; if we use the lower sign, then this quantity will $=0$.

Hence, the above expression will become

$$\frac{2}{3}\pm\frac{n+1}{2n+3}\mp\frac{1}{2}-\frac{1}{2}=\frac{2}{3}-\frac{1}{2}\mp\left(\frac{1}{2}-\frac{n+1}{2n+3}\right)=\frac{1}{6}\mp\frac{1}{2(2n+3)},$$

and since $p=2$, we find

$$S_n=\frac{1}{12}\mp\frac{1}{4(2n+3)};\ S_\infty=\frac{1}{12}.$$

The upper sign has place when n is even, and the lower sign when n is odd.

6. Required the sum of $\frac{4}{1.5}+\frac{4}{5.9}+\frac{4}{9.13}+\frac{4}{13.17}+\text{\&c.}$

Ans. $S_n=1-\frac{1}{4n+1}$; $S_\infty=1$.

7. Required the sum of $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+$ &c.

$$\text{Ans. } S_n=\frac{3}{4}-\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}\right);\ S_\infty=\frac{3}{4}.$$

(200.) Since $\frac{q}{r(r+p)}-\frac{q}{(r+p)(r+2p)}=\frac{2pq}{r(r+p)(r+2p)}$, it follows that

$$\frac{q}{r(r+p)(r+2p)}=\frac{1}{2p}\left(\frac{q}{r(r+p)}-\frac{q}{(r+p)(r+2p)}\right). \qquad \text{(A)}$$

EXAMPLES.

8. Required the sum of $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+$&c.

Comparing these terms with the fraction of the left-hand member of (A), we discover that $p=1$, and $q=4, 5, 6,$ &c., and $r=1, 2, 3,$ &c.

$$\left\{\begin{array}{l}\frac{4}{1.2}+\frac{5}{2.3}+\frac{6}{3.4}+\ldots\ldots\frac{n+3}{n(n+1)} \\ \quad -\left(\frac{4}{2.3}+\frac{5}{3.4}+\ldots\ldots\frac{n+2}{n(n+1)}+\frac{n+3}{(n+1)(n+2)}\right)\end{array}\right\}=$$

$$\frac{4}{1.2}+\left(\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots\frac{1}{n(n+1)}\right)-\frac{n+3}{(n+1)(n+2)}.$$

Now, by example 1, Art. 199, we know that

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}\ldots\ldots\frac{1}{n(n+1)}=\frac{n}{n+1},$$

therefore, the series within the parenthesis

$$=\frac{n}{n+1}-\frac{1}{1.2}.$$

Therefore, we have

$$\frac{4}{1.2}-\frac{1}{1.2}+\frac{n}{n+1}-\frac{n+3}{(n+1)(n+2)}=\frac{3}{2}+\frac{n^2+n-3}{(n+1)(n+2)},$$

which, divided by $2p=2$, gives

$$S_n=\frac{3}{4}+\frac{n^2+n-3}{2(n+1)(n+2)}\,;\ S_\infty=\frac{5}{4}.$$

9. Find the sum of $\frac{1}{1.3.5}+\frac{2}{3.5.7}+\frac{3}{5.7.9}+\frac{4}{7.9.11}+\&c.$

Ans. $S_n=\frac{1}{4}\left(\frac{n}{2n+1}-\frac{n}{(2n+1)(2n+3)}\right);\ S_\infty=\frac{1}{8}.$

10. Find the sum of $\frac{5}{1.2.3}+\frac{6}{2.3.4}+\frac{7}{3.4.5}+\frac{8}{4.5.6}+\&c.$

Ans. $S_n=\frac{n(3n+7)}{2(n+1)(n+2)};\ S_\infty=\frac{3}{2}.$

(201.) It is obvious that this method must be applicable to series, the denominators of whose terms consist of more than three factors; but our limits will not allow us to pursue this subject any further.

RECURRING SERIES.

(202.) *A recurring series* is one, each of whose terms, after a certain number, bears a uniform relation to the same number of those which immediately precede it.

Thus, the series

$$1+2x+8x^2+28a^3+100x^4+356x^5+\ \&c.$$

is a recurring one, each of whose terms, after the first two, can be found by multiplying the next preceding one by $3x$, and the second preceding one by $2x^2$, and taking the sum of the products; thus:

$$8x^2 = 2x \times 3x + 1 \times 2x^2,$$
$$28x^3 = 8x^2 \times 3x + 2x \times 2x^2,$$
$$100x^4 = 28x^3 \times 3x + 8x^2 \times 2x^2,$$
$$356x^5 = 100x^4 \times 3x + 28x^3 \times 2x^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

(203.) The constant multipliers $3x$, $2x^2$, taken together, constitute the *scale of relation*.

Suppose in general

$$A_1 + A_2 + A_3 + A_4 + A_5 + \text{ \&c.},$$

to be a recurring series depending upon the scale of relation p, q, then we shall have

$$\left.\begin{array}{ll} A_1 = A_1, & (1) \\ A_2 = A_2, & (2) \\ A_3 = pA_2 + qA_1, & (3) \\ A_4 = pA_3 + qA_2, & (4) \\ A_5 = pA_4 + qA_3, & (5) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ A_n = pA_{n-1} + qA_{n-2}. & (n) \end{array}\right\} \quad (A)$$

If the scale of relation consist of three parts, p, q, r, we shall have

$$\left.\begin{array}{ll} A_1 = A_1, & (1) \\ A_2 = A_2, & (2) \\ A_3 = A_3, & (3) \\ A_4 = pA_3 + qA_2 + rA_1, & (4) \\ A_5 = pA_4 + qA_3 + rA_2, & (5) \\ A_6 = pA_5 + qA_4 + rA_3, & (6) \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots\cdots\cdots & \\ A_n = pA_{n-1} + qA_{n-2} + rA_{n-3}. & (n) \end{array}\right\} \quad (B)$$

And in a similar way, the successive terms of a recurring series, whose scale of relation consists of more than three parts, can be found.

If we take the sum of the group of equations (A), putting S_n for the sum of n terms of the series, we shall have

$$S_n = A_1 + A_2 + p(S_n - A_1 - A_n) + q(S_n - A_{n-1} - A_n) \qquad (1)$$

This solved for S_n, gives

$$S_n = \frac{(p-1)A_1 - A_2 + qA_{n-1} + (p+q)A_n}{p+q-1}. \qquad \text{(C)}$$

By adding the group (B), and reducing as above, we find

$$S_n = \begin{cases} A_1 + A_2 + A_3 + p(S_n - A_1 - A_2 - A_n) \\ +q(S_n - A_1 - A_{n-1} - A_n) + r(S_n - A_{n-2} - A_{n-1} - A_n) \end{cases}$$

which solved for S_n, gives

$$S_n = \left\{ \begin{array}{l} \dfrac{(p+q-1)A_1 + (p-1)A_2 - A_3}{p+q+r-1} \\ + \dfrac{rA_{n-2} + (q+r)A_{n-1} + (p+q+r)A_n}{p+q+r-1}. \end{array} \right\} \qquad \text{(D)}$$

(204.) Proceeding in this way, we might find similar expressions for S_n when the scale of relation consists of more than three parts.

(205.) If the successive terms of a recurring series are decreasing, and the series is carried to an infinite number of terms, the last terms may be neglected in formulas (C) and (D), as of no appreciable value, therefore, supposing $n = \infty$ in (C) and (D), they become

$$S_\infty = \frac{(p-1)A_1 - A_2}{p+q-1}, \qquad \text{(E)}$$

$$S_\infty = \frac{(p+q-1)A_1 + (p-1)A_2 - A_3}{p+q+r-1}. \qquad \text{(F)}$$

EXAMPLES.

1. What is the sum of the infinite recurring series

$$1+2x+3x^2+4x^3+5x^4+\&c.;$$

the scale of relation being

$$p=2x;\ q=-x^2?$$

In this example $A_1=1$; $A_2=2x$. These values substituted in (E), we find

$$S_\infty=\frac{(2x-1)-2x}{2x-x^2-1}=\frac{1}{(1-x)^2}.$$

2. What is the sum of the infinite recurring series

$$1+2x+8x^2+28x^3+100x^4+\&c;$$

the scale of relation being

$$p=3x;\ q=2x^2?$$

We also have in this example $A_1=1$; $A_2=2x$, which values cause (E) to become

$$S_\infty=\frac{1-x}{1-2x-2x^2}.$$

3. What is the sum of the infinite recurring series

$$1+x+5x^2+13x^3+41x^4+\&c.;$$

the scale of relation being

$$p=2x;\ q=3x^2?$$

Ans. $S_\infty=\dfrac{1-x}{1-2x-3x^2}$.

4. What is the sum of the infinite recurring series

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+\&c.;$$

in which the scale of relation is

$$p=x;\ q=x^2;\ r=-x^3?$$

Ans. $S_\infty=\dfrac{1}{1-x-x^2+x^3}$.

5. What is the sum of the infinite recurring series

$$1+4x+6x^2+11x^3+28x^4+63x^5+ \text{\&c.};$$

in which the scale of relation is

$$p=2x;\ q=-x^2;\ r=3x^3\,?$$

$$\text{Ans. } S_\infty=\frac{(1+x)^2-2x^2}{(1+x)^2-3x^3}.$$

From these examples we see that the sum of an infinite number of terms of a converging recurring series, is in the form of a rational fraction. Conversely, all rational fractions, when expanded by actual division, or by the method of indeterminate coefficients, as accomplished under Art. 184, will give a recurring series.

(206.) When the scale of relation is not given, it may be found by means of a few of the first terms of the series, thus:

Resuming our general equation (n) of group (A) Art. 203, where the scale of relation consists of two parts, p and q, we have

$$A_n=pA_{n-1}+qA_{n-2}. \qquad (1)$$

Writing $n+1$ for n, it becomes

$$A_{n+1}=pA_n+qA_{n-1}. \qquad (2)$$

From these two equations we readily deduce

$$p=\frac{A_{n+1}A_{n-2}-A_nA_{n-1}}{A_nA_{n-2}-A^2_{n-1}}, \qquad (3)$$

$$q=-\frac{A_{n+1}A_{n-1}-A^2_n}{A_nA_{n-2}-A^2_{n-1}}. \qquad (4)$$

If in (3) and (4) we put $n=3$, they will become

$$p=\frac{A_4A_1-A_3A_2}{A_3A_1-A^2_2}, \qquad (5)$$

$$q=-\frac{A_4A_2-A^2_3}{A_3A_1-A^2_2}. \qquad (6)$$

From (5) and (6), we shall be able to find the scale of relation, when it consists of but two parts, by the aid of the first four terms of the series. Equations (3) and (4) show that the scale may be found by using any four consecutive terms.

By a similar process we might, by the aid of the first six terms of the series, find the scale of relation when it consists of three parts.

(207.) *A geometrical series may be also a recurring series.*

To prove this, we will take the general form of a geometrical series (178).

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \dots\dots\dots \quad (1)$$

Now, in order that this may be a recurring series, having p and q for the scale of relation, we must have, (103),

$$\left.\begin{aligned} ar^2 &= par + qa, \\ ar^3 &= par^2 + qar, \\ ar^4 &= par^3 + qar^2, \\ ar^5 &= par^4 + qar^3, \\ &\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots \\ &\dots\dots\dots\dots\dots \\ ar^n &= par^{n-1} + qar^{n-2}. \end{aligned}\right\} \quad (2)$$

By striking out the factors common to these conditions, we see that each becomes

$$r^2 = pr + q, \quad (3)$$

which gives

$$r = \frac{p}{2} \pm \tfrac{1}{2}\sqrt{p^2 + 4q}. \quad (4)$$

When these two values of r are real, and not equal, there will be two geometrical series which will also be recurring, having p and q for the scale of relation.

We will denote these two values of r, by r' and r''. And since it is immaterial what value we take for a, the first term, we will take a' for the first term when we use r', and a'' when we use r''. The two series will then become

$$\left.\begin{array}{l} a'+a'r'+a'r'^2+a'r'^3+a'r'^4+a'r'^5+\ldots, \\ a''+a''r''+a''r''^2+a''r''^3+a''r''^4+a''r''^5+\ldots, \end{array}\right\} \quad (5)$$

each of which is a recurring series having p and q for the scale of relation. If we take the sum of the corresponding terms of the two series (5), we shall find

$$\left.\begin{array}{l} (a'+a'')+(a'r'+a''r'')+(a'r'^2+a''r''^2) \\ +(a'r'^3+a''r''^3)+(a'r'^4+a''r''^4)+\&c., \end{array}\right\} \quad (6)$$

which is not a geometrical series, but is, nevertheless, a recurring series having p and q for the scale of relation.

In all recurring series whose scale of relation consists of two parts, we must, in order to be able to compute the successive terms, know the values of the first two terms, which we have represented by A_1 and A_2.

Since the values of a' and a'', in (6), have not yet been fixed, it is evident they may be so taken as to make the first two terms of (6) agree with A_1 and A_2. This is effected by making

$$a'+a''=A_1, \quad (7)$$

$$a'r'+a''r''=A_2. \quad (8)$$

These equations immediately give

$$a'=\frac{A_2-A_1r''}{r'-r''}, \quad (9)$$

$$a''=-\frac{A_2-A_1r'}{r'-r''}. \quad (10)$$

The nth term, A_n, of the recurring series (6) is

$$a'r'^{n-1}+a''r''^{n-1}.$$

Hence, if we substitute the values of a' and a'', as given by (9) and (10), we shall have

$$A_n=\frac{A_2-A_1r''}{r'-r''}\cdot r'^{n-1}-\frac{A_2-A_1r'}{r'-r''}\cdot r''^{n-1}. \quad \text{(A)}$$

(208.) By a similar train of reasoning, we might show that a recurring series whose scale of relation consists of three parts, is the sum of three *geometrical recurring series*, having the same scale of relation.

EXAMPLES.

1. The recurring series $1+x+3x^2+5x^3+$ &c., whose scale of relation is x and $2x^2$, is the sum of the two following *geometrical recurring series*, each having the same scale of relation:

$$\frac{2}{3}+\frac{4}{3}x+\frac{8}{3}x^2+\frac{16}{3}x^3+\frac{32}{3}x^4+\text{\&c.}$$
$$\frac{1}{3}-\frac{1}{3}x+\frac{1}{3}x^2-\frac{1}{3}x^3+\frac{1}{3}x^4-\text{\&c.}$$
$$1+x+3x^2+5x^3+11x^4+\text{\&c.}$$

2. The recurring series, $1+x+5x^2+13x^3+$&c., whose scale of relation is $2x$ and $3x^2$, is the sum of the two following *geometrical recurring series*, each having the same scale of relation:

$$\frac{1}{2}+\frac{3}{2}x+\frac{9}{2}x^2+\frac{27}{2}x^3+\frac{81}{2}x^4+\text{\&c.}$$
$$\frac{1}{2}-\frac{1}{2}x+\frac{1}{2}x^2-\frac{1}{2}x^3+\frac{1}{2}x^4-\text{\&c.}$$
$$1+x+5x^2+13x^3+41x^4+\text{\&c.}$$

3. The recurring series, $1+x+2x^2+3x^3+6x^4+$&c., whose scale of relation is $2x$, x^2, and $-2x^3$, is the sum of the three following *geometrical recurring series*, each having the same scale of relation :

$$\begin{array}{l}
\frac{1}{2}+\frac{1}{2}x+\frac{1}{2}x^2+\frac{1}{2}x^3+\frac{1}{2}x^4+\text{\&c.}\\
\frac{1}{3}+\frac{2}{3}x+\frac{4}{3}x^2+\frac{8}{3}x^3+\frac{16}{3}x^4+\text{\&c.}\\
\frac{1}{6}-\frac{1}{6}x+\frac{1}{6}x^2-\frac{1}{6}x^3+\frac{1}{6}x^4-\text{\&c.}\\
\hline
1+\ x+2x^2+3x^3+6x^4+\text{\&c.}
\end{array}$$

(209.) From what has been shown, it follows that we may regard all geometrical series as recurring series, but all recurring series are not geometrical series. When a recurring series is not a geometrical series, it is the sum of two or more geometrical series. Hence, a geometrical series may be regarded as a particular case of a recurring series, the recurring series being of a more general form.

(210.) We will now give some examples in which the general term, $\mathcal{A}_n$, of a recurring series is required.

EXAMPLES.

1. Suppose the scale of relation of the series

$$1+x+3x^2+5x^3+11x^4+21x^5+\text{\&c.},$$

to be $p=x$; $q=2x^2$, what will be the nth term?

In formula (A), the values of r' and r'' are given by (4). In this example they become

$$r'=2x;\ r''=-x.$$

From the given series we have

$$\mathcal{A}_1=1;\ \mathcal{A}_2=x.$$

Substituting these values in formula (A), we find the

Ans. $A_n = \frac{2}{3}(2x)^{n-1} + \frac{1}{3}(-x)^{n-1}$.

2. If the scale of relation of the series

$$1 + x + 5x^2 + 13x^3 + 41x^4 + 121x^5 + \&c.,$$

is $p = 2x$; $q = 3x^2$, what will be the nth term?

Ans. $A_n = \frac{1}{2}(3x)^{n-1} + \frac{1}{2}(-x)^{n-1}$.

3. The scale of relation of the series

$$1 + 2x + 5x^2 + 13x^3 + 34x^4 + 89x^5 + \&c.,$$

being $p = 3x$; $q = -x^2$, what will be the nth term?

$$\text{Ans. } A_n = \begin{cases} \dfrac{(\sqrt{5}+1)(3+\sqrt{5})^{n-1}}{2^n\sqrt{5}} . x^{n-1} \\ + \dfrac{(\sqrt{5}-1)(3-\sqrt{5})^{n-1}}{2^n\sqrt{5}} . x^{n-1}. \end{cases}$$

Referring to the question of the oak tree, under Chap. XII, Higher Arithmetic, we see that if we call $x = 1$, the above expression for A_n will give the number of branches of the tree, at the end of n years, thus the number of branches at the end of 20 years is

$$\frac{(1+\sqrt{5})(3+\sqrt{5})^{19}}{2^{20}\sqrt{5}} - \frac{(1-\sqrt{5})(3-\sqrt{5})^{19}}{2^{20}\sqrt{5}}.$$

(211.) Having shown how to find the general term of a recurring series, it is easy to find the sum of n terms by the aid of formulas (C) and (D), Art. 203.

EXAMPLES.

1. Find the sum of n terms of the recurring series

$$1 + x + 3x^2 + 5x^3 + 11x^4 + 21x^5 + \&c.,$$

whose scale of relation is

$$p = x; \; q = 2x^2.$$

Under the last Article, we have found the nth term to be

$$A_n = \frac{2}{3}(2x)^{n-1} + \frac{1}{3}(-x)^{n-1}.$$

Writing $n-1$ for n, we find

$$A_{n-1} = \frac{2}{3}(2x)^{n-2} + \frac{1}{3}(-x)^{n-2}.$$

Substituting these values of A_n and A_{n-1}, together with the known values of p and q, in (C), Art. 203, we have

$$S_n = \frac{2^{n+1}(1+x)x^n + (1-2x)(-x)^n - 3}{3(2x^2+x-1)},$$

or, perhaps a simpler form is

$$S_n = \frac{(2^{n+1} \mp 2)x^{n+1} + (2^{n+1} \pm 1)x^n - 3}{6x^2 + 3x - 3}.$$

In this last expression, the upper sign is to be used when n is *even*.

2. Find the sum of n terms of the recurring series

$$1 + x + 5x^2 + 13x^3 + 41x^4 + \text{\&c.},$$

whose scale of relation is

$$p = 2x;\ q = 3x^2.$$

In this example, the nth term is

$$A_n = \frac{1}{2}(3x)^{n-1} + \frac{1}{2}(-x)^{n-1},$$

or which is the same thing,

$$A_n = \frac{1}{2}(3^{n-1} \mp 1)x^{n-1}.$$

The upper sign must be used when n is *even*. Writing $n-1$ for n, we find

$$A_{n-1} = \frac{1}{2}(3^{n-2} \mp 1)x^{n-2}.$$

The values of A_n, A_{n-1}, p, and q, being used, cause formula (C) to become

$$S_n = \frac{3(3^{n-1} \mp 1)x^{n+1} + (3^n \pm 1)x^n + 2x - 2}{6x^2 + 4x - 2}.$$

Use the upper sign when n is *even*.

CHAPTER VIII.

CONTINUED FRACTIONS.

(212.) Suppose we have the following conditions:

$$A = y + \frac{x_1}{D_1}, \qquad (1)$$

$$D_1 = y_1 + \frac{x_2}{D_2}, \qquad (2)$$

$$D_2 = y_2 + \frac{x_3}{D_3}, \qquad (3)$$

$$D_3 = y_3 + \frac{x_4}{D_4}, \qquad (4)$$

&c. &c.

In (1), for D_1, write its value as given by (2), and it becomes $A = y + \cfrac{x_1}{y_1 + \cfrac{x_2}{D_2}}$. In this expression, for D_2, write its value given by (3), and we have

$$A = y + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \cfrac{x_3}{D_3}}}.$$

Now substituting for D_3, its value given by (4), and we obtain

$$A = y + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \cfrac{x_3}{y_3 + \cfrac{x_4}{D_4 + \text{\&c.}}}}} \ldots\ldots \quad (5)$$

(213.) Such expressions as the above value of A, equation (5), are called CONTINUED FRACTIONS.

The expressions $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \frac{x_3}{y_3}$, &c., of which $\frac{x_n}{y_n}$ is the general term, we shall call *partial fractions*. The numerators $x_1, x_2\, x_3, \ldots\ldots x_n$, we shall call *partial numerators*.

The denominators $y_1, y_2, y_3, \ldots\ldots y_n$, in like manner, we shall call *partial denominators*.

If we compute successively 1, 2, 3, 4, &c., terms of the continued fraction (5), by reducing them to the form of common fractions, we shall find

$$y \ldots\ldots = \frac{y}{1} \ldots\ldots = \frac{p_1}{q_1}.$$

$$y + \frac{x_1}{y_1} \ldots\ldots = \frac{yy_1 + x_1}{y_1} \ldots\ldots = \frac{p_2}{q_2}.$$

$$y + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2}} \ldots\ldots = \frac{yy_1y_2 + x_1y_2 + yx_2}{y_1y_2 + x_2} \ldots\ldots = \frac{p_3}{q_3}.$$

$$y + \cfrac{x_1}{y_1 + \cfrac{x_2}{y_2 + \cfrac{x_3}{y_3}}} = \frac{yy_1y_2y_3 + x_1y_2y_3 + yx_2y_3 + yy_1x_3 + x_1x_3}{y_1y_2y_3 + x_2y_3 + y_1x_3} = \frac{p_4}{q_4}.$$

&c. .... &c.

We shall hereafter represent these successive values, which we shall call *approximative fractions*, by the abridged expressions $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}$, &c., of which the general term is $\frac{p_n}{q_n}$.

By carefully examining the above approximative fractions, we discover the following relations:

$p_3 = p_2y_2 + p_1x_2$, (6) $\qquad q_3 = q_2y_2 + q_1x_2$, (8)

$p_4 = p_3y_3 + p_2x_3$, (7) $\qquad q_4 = q_3y_3 + q_2x_3$. (9)

By referring to our continued fraction (5), we see that $\frac{p_4}{q_4}$ will change to $\frac{p_5}{q_5}$, if we substitute $y_3 + \frac{x_4}{y_4}$ for y_3. Making this substitution in (7) and (9), we find

$$\frac{p_5}{q_5} = \frac{p_3\left(y_3 + \frac{x_4}{y_4}\right) + p_2x_3}{q_3\left(y_3 + \frac{x_4}{y_4}\right) + q_2x_3} = \frac{y_4(p_3y_3 + p_2x_3) + p_3x_4}{y_4(q_3y_3 + q_2x_3) + q_3x_4}$$

$$= \frac{p_4y_4 + p_3x_4}{q_4y_4 + q_3x_4}.$$

Hence, $p_5 = p_4y_4 + p_3x_4$, (10) $\qquad q_5 = q_4y_4 + q_3x_4$. (11)

And in the same way may p_6 and q_6 be drawn from the next two inferior values. Therefore the law is general and may be expressed as follows:

$$p_n = p_{n-1}y_{n-1} + p_{n-2}x_{n-1}, \qquad (12)$$

$$q_n = q_{n-1}y_{n-1} + q_{n-2}x_{n-1}. \qquad (13)$$

(214.) If we place our quantities in the following order:

$$\begin{array}{cccccccc}
y & y_1 & y_2 & y_3 & y_4 & \ldots\ldots\ldots\ldots & y_{n-1} & y_n \\
\frac{1}{0}; & \frac{y}{1}; & \frac{p_2}{q_2}; & \frac{p_3}{q_3}; & \frac{p_4}{q_4}; & \ldots\ldots\ldots\ldots & \frac{p_{n-1}}{q_{n-1}}; & \frac{p_n}{q_n}; \&c. \\
x_1 & x & x_3 & x_4 & x_5 & & x_n & x_{n+1}
\end{array}$$

We may find the successive approximative fractions by the following

RULE.

Multiply the numerator of the last approximative fraction by the partial denominator which stands over it, and the numerator of the approximative fraction which precedes this, by the partial numerator which stands under it; the sum of these products, noticing the signs, is the numerator of the next approximative fraction. In like manner we must multiply the denominator of the last approximative fraction by the partial denominator which is over it, and the denominator of the approximative fraction which precedes this by the partial numerator which stands under it; the sum of these products is the denominator of the next approximative fraction.

EXAMPLES.

1. Find, by the above rule, some of the approximative fractions of the infinite continued fraction

$$a-\cfrac{a^2}{3-\cfrac{a^2}{5-\cfrac{a^2}{7-\cfrac{a^2}{9-\&c.}}}}$$

Our work, when executed agreeable to the above rule, will be as follows:

$$\begin{array}{cccccc}
a & 3 & 5 & 7 & 9 & \\
\dfrac{1}{0}, & \dfrac{a}{1}, & \dfrac{3a-a^2}{3}, & \dfrac{15a-5a^2-a^3}{15-a^2}, & \dfrac{105a-35a^2-10a^3+a^4}{105-10a^2}, & \\
-a^2 & -a^2 & -a^2 & -a^2 & -a^2 & [\&c.
\end{array}$$

2. Find some of the approximate fractions of the continued fraction

$$\cfrac{a}{1+\cfrac{2a}{1+\cfrac{3a}{1+\cfrac{4a}{1+\cfrac{5a}{1+\&c.}}}}}$$

In this example, y, the integral part, is nothing, and our work is as follows:

$$\begin{array}{cccccc} 0 & 1 & 1 & 1 & 1 & 1 \\ \frac{1}{0}, & \frac{0}{1}, & \frac{a}{1}, & \frac{a}{1+2a}, & \frac{a+3a^2}{1+5a}, & \frac{a+7a^2}{1+9a+8a^2}, \&c. \\ a & 2a & 3a & 4a & 5a & 7a \end{array}$$

(215.) If, in our general expression (5), we suppose all the partial denominators $y_1, y_2, y_3, \ldots\ldots y_n$ to be positive, and also $1 = x_1 = x_2 = x_3 = \ldots\ldots = x_n$, we shall then have

$$A = y + \cfrac{1}{y_1 + \cfrac{1}{y + \cfrac{1}{y_3 + \cfrac{1}{y_4 + \&c.}}}} \qquad (14)$$

This is the kind of continued fraction most commonly employed. Any common vulgar fraction can be converted into a continued fraction of the above form, by the method explained in my *Higher Arithmetic*, which is equivalent to the following

RULE.

Divide the denominator by the numerator; then divide this divisor by the remainder, and thus continue to divide the preceding divisor by the last remainder, until there is no remainder, or until we have obtained as many terms as we

wish; then will these successive quotients be respectively the values of y_1, y_2, y_3, ... y_n.

NOTE.—In this rule we have supposed the vulgar fraction to be less than a unit, and consequently the integral part $y = 0$; when the fraction is not less than 1, we may first reduce it to a mixed number, and then proceed with the fractional part agreeable to the above rule.

EXAMPLES.

1. Convert $\frac{516901}{740785}$ into a continued fraction.

OPERATION.

$$
\begin{array}{l}
516901)740785(1 = y_1 \\
\quad 516901 \\
\quad \overline{223884)516901(2 = y_2} \\
\qquad 447768 \\
\qquad \overline{69133)223884(3 = y_3} \\
\qquad\quad 207399 \\
\qquad\quad \overline{16485)69133(4 = y_4} \\
\qquad\qquad 65940 \\
\qquad\qquad \overline{3193)16485(5 = y_5} \\
\qquad\qquad\quad 15965 \\
\qquad\qquad\quad \overline{520)3193(6 = y_6} \\
\qquad\qquad\qquad 3120 \\
\qquad\qquad\qquad \overline{73)520(7 = y_7} \\
\qquad\qquad\qquad\quad 511 \\
\qquad\qquad\qquad\quad \overline{9)73(8 = y_8} \\
\qquad\qquad\qquad\qquad 72 \\
\qquad\qquad\qquad\qquad \overline{1)9(9 = y_9} \\
\qquad\qquad\qquad\qquad\quad 9 \\
\qquad\qquad\qquad\qquad\quad \overline{0}
\end{array}
$$

Therefore, $\frac{516901}{740785}=\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8+\cfrac{1}{9}}}}}}}}}.$

2. The fraction $\frac{10000000000}{31415926535}$ expresses nearly the ratio of the diameter of a circle to its circumference; required to expand it into a continued fraction.

Proceeding as above, we find

$$\frac{10000000000}{31415926535}=\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{15+\cfrac{1}{1+\cfrac{1}{292+\&c.}}}}}$$

(216.) Referring to (14), we see that $A>y$, since, in order to obtain the true value of A, something must be added to y. Again, $A<y+\frac{1}{y_1}$, since, in order to obtain the true value of A, the denominator y_1 must be increased, and consequently $y+\frac{1}{y_1}$ will be diminished. We can show in the same way that

$$A>y+\cfrac{1}{y_1+\cfrac{1}{y_2}}, \qquad A<y+\cfrac{1}{y+\cfrac{1}{y_2+\cfrac{1}{y_3}}}.$$

(217.) *Therefore, the value of A is always comprised between two consecutive approximative fractions.*

(218.) When $1 = x_1 = x_2 = x_3 = \ldots\ldots = x_n$, equations (12) and (13) become

$$p_n = p_{n-1}y_{n-1} + p_{n-2}, \tag{15}$$

$$q_n = q_{n-1}y_{n-1} + q_{n-2}. \tag{16}$$

Multiplying (15) by q_{n-1}, and (16) by p_{n-1}, and then taking the difference of the results, we find

$$p_nq_{n-1} - p_{n-1}q_n = -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}). \tag{17}$$

This shows that $p_nq_{n-1} - p_{n-1}q_n$ and $p_{n-1}q_{n-2} - p_{n-2}q_{n-1}$ are equal in numerical value, but contrary in sign. When $n = 2$, equation (17) gives

$$p_2q_1 - p_1q_2 = -(p_1q_0 - p_0q_1).$$

We know that $\frac{p_1}{q_1} = \frac{y}{1}$ and $\frac{p_0}{q_0} = \frac{1}{0}$; consequently, $p_1q_0 - p_0q_1 = -1$; therefore, $p_2q_1 - p_1q_2 = 1$, $p_3q_2 - p_2q_3 = -1$, $p_4q_3 - p_3q_4 = 1$; and so on for other similar expressions.

Hence, we always have

$$p_nq_{n-1} - p_{n-1}q_n = \pm 1. \tag{18}$$

The upper sign having place when n is *even*, and the lower sign when n is *odd*.

If we take the difference of two consecutive approximative fractions we shall find

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{p_nq_{n-1} - p_{n-1}q_n}{q_nq_{n-1}}.$$

By (18), we know that the numerator of the right-hand member of this equation is $= \pm 1$. Hence,

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{\pm 1}{q_nq_{n-1}}. \tag{19}$$

This shows, that the difference between any two consecutive approximative fractions is equal to the reciprocal of the product of their denominators.

We have already shown that the true value lies between the values of any two consecutive approximative fractions, and since $q_n > q_{n-1}$, we have $\frac{1}{q_{n-1}^2} > \frac{1}{q_{n-1}\,q_n}$. Therefore the difference between $\frac{p_{n-1}}{q_{n-1}}$ and A is less than $\frac{1}{q_{n-1}^2}$. *That is, the difference between the true value and any approximative fraction, is less than the reciprocal of the square of its denominator.*

Dividing (15) by (16) we find

$$\frac{p_n}{q_n} = \frac{p_{n-1}y_{n-1} + p_{n-2}}{q_{n-1}y_{n-1} + q_{n-2}}. \qquad (20)$$

If, in this equation, for y_{n-1}, we substitute the complete denominator, which we will represent by z, then will $\frac{p_n}{q_n} = A$. From the form of our continued fraction (14), it is obvious that z will also be in the form of a continued fraction.

Thus, $z = y_{n-1} + \cfrac{1}{y_n + \cfrac{1}{y_{n+1} + \cfrac{1}{y_{n+2} + \&c.}}}$

This value of z being substituted for y_{n-1} in (20), we have

$$A = \frac{p_{n-1}z + p_{n-2}}{q_{n-1}z + q_{n-2}}. \qquad (21)$$

Equation (21) immediately gives

$$A - \frac{p_{n-1}}{q_{n-1}} = \frac{p_{n-2}\,q_{n-1} - p_{n-1}\,q_{n-2}}{q_{n-1}(q_{n-1}z + q_{n-2})}. \qquad (22)$$

$$A - \frac{p_{n-2}}{q_{n-2}} = \frac{(p_{n-1}\,q_{n-2} - p_{n-2}\,q_{n-1})z}{q_{n-2}(q_{n-1}z + q_{n-2})}. \qquad (23)$$

The condition of (18) causes these to become

$$\mathcal{A}-\frac{p_{n-1}}{q_{n-1}}=\frac{\mp 1}{q_{n-1}(q_{n-1}z+q_{n-2})}, \qquad (24)$$

$$\mathcal{A}-\frac{p_{n-2}}{q_{n-2}}=\frac{\pm z}{q_{n-2}(q_{n-1}z+q_{n-2})}. \qquad (25)$$

Now, by the nature of continued fractions, $z>1$, and $q_{n-1}>q_{n-2}$. Hence, the right-hand member of (25) is greater than the right-hand member of (24). Considering the numerical value without reference to the signs. *This shows, that each approximate fraction is nearer the true value than the preceding approximative fraction.*

If $\frac{p_{n-2}}{q_{n-2}}>\mathcal{A}$, conditions (24) and (25) will give

$$\mathcal{A}=\frac{p_{n-1}}{q_{n-1}}+\frac{1}{q_{n-1}(q_{n-1}z+q_{n-2})}, \qquad (26)$$

$$\mathcal{A}=\frac{p_{n-2}}{q_{n-2}}-\frac{z}{q_{n-2}(q_{n-1}z+q_{n-2})}. \qquad (27)$$

If $\frac{p_{n-2}}{q_{n-2}}<\mathcal{A}$, conditions (24) and (25) will give

$$\mathcal{A}=\frac{p_{n-1}}{q_{n-1}}-\frac{1}{q_{n-1}(q_{n-1}z+q_{n-2})}. \qquad (28)$$

$$\mathcal{A}=\frac{p_{n-2}}{q_{n-2}}+\frac{z}{q_{n-2}(q_{n-1}z+q_{n-2})}. \qquad (29)$$

Equations (26) and (27) give

$$\frac{1}{2}\left\{\frac{p_{n-2}}{q_{n-2}}+\frac{p_{n-1}}{q_{n-1}}\right\}>\mathcal{A}. \qquad (30)$$

Equations (28) and (29) give

$$\frac{1}{2}\left\{\frac{p_{n-2}}{q_{n-2}}+\frac{p_{n-1}}{q_{n-2}}\right\}<\mathcal{A}. \qquad (31)$$

Hence, the successive arithmetical means of two consecutive approximative fractions are alternately greater and less than the true value.

If we take the product of (26) and (27), we find, after a little reduction,

$$\left.\begin{array}{l} A^2=\dfrac{p_{n-2}p_{n-1}}{q_{n-2}q_{n-1}} \\ -\dfrac{(p_{n-1}q_{n-1}z-p_{n-2}q_{n-2})+(p_{n-1}q_{n-2}-p_{n-2}q_{n-1})z+z}{q_{n-2}q_{n-1}(q_{n-1}z+q_{n-2})^2}. \end{array}\right\} \quad (32)$$

Now, since $\dfrac{p_{n-2}}{q_{n-2}}>A$, we have, by (18),

$$(p_{n-1}q_{n-2}-p_{n-2}q_{n-1})z+z=0.$$

Moreover, we have $p_{n-1}q_{n-1}z>p_{n-2}q_{n-2}$.

Consequently, $\dfrac{p_{n-2}p_{n-1}}{q_{n-2}q_{n-1}}>A^2$. (33)

Taking the product of (28) and (29), we find

$$\left.\begin{array}{l} A^2=\dfrac{p_{n-2}p_{n-1}}{q_{n-2}q_{n-1}} \\ +\dfrac{(p_{n-1}q_{n-1}z-p_{n-2}q_{n-2})+(p_{n-1}q_{n-2}-p_{n-2}q_{n-1})z+z}{q_{n-2}q_{n-1}(q_{n-1}z+q_{n-2})^2}. \end{array}\right\} \quad (34)$$

Now, since $\dfrac{p_{n-2}}{q_{n-2}}<A$, we have, by (18),

$$(p_{n-1}q_{n-2}-p_{n-2}q_{n-1})z-z=0.$$

And, as before, $p_{n-1}q_{n-1}z>p_{n-2}q_{n-2}$.

Consequently, $\dfrac{p_{n-2}p_{n-1}}{q_{n-2}q_{n-1}}<A^2$. (35)

Equations (33) *and* (35) *show that the successive geometrical means of two consecutive approximative fractions are alternately greater and less than the true value.*

(219.) All approximative fractions are always in their lowest terms. For if not, let the numerator p_n, and its

denominator q_n, of the approximative fraction $\frac{p_n}{q_n}$, have a common divisor h.

Condition (18) shows, that if p_n and q_n are each divisible by h, then its left-hand member must be divisible by h, and consequently its right-hand member is also divisible by h; that is, ± 1 is divisible by h, which is absurd; consequently it is absurd to suppose p_n and q_n to be divisible by h.

(220.) In the case of $1 = x_1 = x_2 = x_3 = \ldots\ldots = x_n$, the rule under (Art. 214) will require some modification in order to appear in its simplest form. Thus, placing the quantities as follows:

$$\begin{array}{cccccccc} y & y_1 & y_2 & y_3 & y_4 \ldots\ldots & y_{n-1} & y_n \ldots\ldots & \\ \frac{1}{0}, & \frac{y}{1}, & \frac{p_2}{q_2}, & \frac{p_3}{q_3}, & \frac{p_4}{q_4} \ldots\ldots & \frac{p_{n-1}}{q_{n-1}}, & \frac{p_n}{q_n}, & \&c. \end{array}$$

We deduce the successive approximate values by this

RULE.

Multiply the numerator of the last approximative fraction by the partial denominator standing over it, and to the product add the numerator of the preceding approximative fraction, and the sum will be the numerator of the next approximative fraction. In like manner multiply the denominator of the last approximative fraction by the partial denominator standing over it, and to the product add the denominator of the preceding approximative fraction, and it will give the denominator of the next approximative fraction.

EXAMPLES.

1. What are some of the approximative fractions of the infinite continued fraction

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{6+\cfrac{1}{7+\cfrac{1}{8+\cfrac{1}{9}\,\&c.?}}}}}}}}$$

Ans. $\frac{1}{1}, \frac{2}{3}, \frac{7}{10}, \frac{30}{43}, \frac{157}{225}, \frac{972}{1393}$ &c.

2. What are the approximative fractions of the continued fraction $\cfrac{1}{3+\cfrac{1}{7+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{2}}}}}}}}$?

Ans. $\frac{1}{3}, \frac{7}{22}, \frac{8}{25}, \frac{23}{72}, \frac{100}{313}, \frac{523}{1637}, \frac{623}{1950}, \frac{1769}{5537}$.

This last value expresses accurately the true value of the above continued fraction. Whenever the value of a continued fraction is capable of being expressed rationally, it must consist of a finite number of terms; but when the value is irrational, the continued fraction will extend to infinity.

(221.) Continued fractions may be employed for determining approximately the values of the square roots of surds.

Operating upon $\sqrt{19}$, we obtain the successive values:

$$A=\sqrt{19}=4+\frac{\sqrt{19}-4}{1}=y+\frac{1}{D_1}.$$

$$D_1=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=y_1+\frac{1}{D_2}.$$

$$D_2=\frac{3}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}=y_2+\frac{1}{D_3}.$$

$$D_3=\frac{5}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}=y_3+\frac{1}{D_4}.$$

$$D_4=\frac{2}{\sqrt{19}-3}=\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}=y_4+\frac{1}{D_5}.$$

$$D_5=\frac{5}{\sqrt{19}-2}=\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}=y_5+\frac{1}{D_6}.$$

$$D_6=\frac{3}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{1}=8+\frac{\sqrt{19}-4}{1}=y_6+\frac{1}{D_7}.$$

$$D_7=\frac{1}{\sqrt{19}-4}=\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=y_7+\frac{1}{D_8}.$$

&c...&c.

Collecting the results, we have

$y=4$, $y_1=2$, $y_2=1$, $y_3=3$, $y_4=1$, $y_5=2$, $y_6=8$, $y_7=2$, &c.

$D_1=\frac{\sqrt{19}+4}{3}$, $D_2=\frac{\sqrt{19}+2}{5}$, $D_3=\frac{\sqrt{19}+3}{2}$, $D_4=\frac{\sqrt{19}+3}{5}$, $D_5=\frac{\sqrt{19}+2}{3}$, $D_6=\frac{\sqrt{19}+4}{1}$, $D_7=\frac{\sqrt{19}+4}{3}$.

The values of D_1, D_2, D_3, &c., of which the general term is D_n, are *complete denominators* of their corresponding partial fractions.

The values of y_1, y_2, y_3, &c., which we have already called *partial denominators*, are the greatest integral parts of

their corresponding complete denominators D_1, D_2, D_3, &c. In this example, we see that $D_7 = D_1 = \frac{\sqrt{19}+4}{3}$, and $y_7 = y_1 = 2$, hence the operation must begin to repeat at this point, and the partial denominators as well as the complete denominators will recur in an infinite number of periods.

(222.) Suppose we wish the value of the surd $\sqrt{B}$.

If a^2 is the greatest square contained in B, with the remainder b, we shall obviously have

$$\left.\begin{aligned} \sqrt{B} &= a + \frac{\sqrt{B}-a}{1}, \\ D_1 &= \frac{1}{\sqrt{B}-a} = \frac{\sqrt{B}+a}{B-a^2} = \frac{\sqrt{B}+a}{b}. \end{aligned}\right\} \quad (36)$$

The form of the general value of D_n will be

$$D_n = \frac{\sqrt{B}+M_n}{N_n}. \quad (37)$$

If $n+1$ is written for n, this becomes

$$D_{n+1} = \frac{\sqrt{B}+M_{n+1}}{N_{n+1}}. \quad (38)$$

Now, by carefully inspecting the operations just performed for finding the value of $\sqrt{19}$, (Art. 221), we draw this relation:

$$D_n = y_n + \frac{1}{D_{n+1}}. \quad (39)$$

Substituting in (39) the values of D_n, D_{n+1}, given by (37) and (38), we find

$$\frac{\sqrt{B}+M_n}{N_n} = y_n + \frac{N_{n+1}}{\sqrt{B}+M_{n+1}}. \quad (40)$$

This cleared of fractions, becomes

$$\left.\begin{array}{l} B+(M_n+M_{n+1})\sqrt{B}+M_nM_{n+1} \\ =y_nN_n\sqrt{B}+y_nN_nM_{n+1}+N_nN_{n+1}. \end{array}\right\}$$

Equating the irrational parts, as well as the rational parts (Art. 116), we find

$$M_n+M_{n+1}=y_nN_n, \qquad (41)$$

$$B+M_nM_{n+1}=y_nN_nM_{n+1}+N_nN_{n+1}. \qquad (42)$$

These equations readily give

$$M_{n+1}=y_nN_n-M_n. \qquad (43)$$

$$N_{n+1}=\frac{B-M^2_{n+1}}{N_n}. \qquad (44)$$

If in (44) we substitute for M_{n+1} its value given by (43), we find

$$N_{n+1}=\frac{B-M_n^2}{N_n}-y_n^2N_n+2y_nM_n. \qquad (45)$$

Equation (44) gives

$$N_n=\frac{B-M_n^2{}_{+1}}{N_{n+1}}. \qquad (46)$$

In (46) writing $n-1$ for n, we get

$$N_{n-1}=\frac{B-M_n^2}{N_n}. \qquad (47)$$

This causes (45) to become

$$N_{n+1}=N_{n-1}-y_n^2N+2y_nM_n. \qquad (48)$$

Equations (43) and (48) show, that when N_{n-1}, N_n, and M_n are whole numbers, then will N_{n+1} and M_{n+1} be whole numbers. When $n=1$, we have by (36) $N_{n-1}=1$, $N_n=b$, and $M_n=a$.

Hence, N_n and M_n are whole numbers for all values of n.

(223.) If in (21) we substitute $\dfrac{\sqrt{B}+M_{n-1}}{N_{n-1}}$ for z, A will change to $\sqrt{B}$, and we shall have

$$\sqrt{B}=\frac{p_{n-1}\left(\frac{\sqrt{B}+M_{n-1}}{N_{n-1}}\right)+p_{n-2}}{q_{n-1}\left(\frac{\sqrt{B}+M_{n-1}}{N_{n-1}}\right)+q_{n-2}}. \qquad (49)$$

Clearing this of fractions and reducing, we have

$$\left.\begin{aligned}q_{n-1}B+q_{n-1}M_{n-1}\sqrt{B}+q_{n-2}N_{n-1}\sqrt{B}=p_{n-1}\sqrt{B}\\ +p_{n-1}M_{n-1}+p_{n-2}N_{n-1}.\end{aligned}\right\}$$

Equating the rational quantities, as well as the irrational, we have

$$p_{n-1}M_{n-1}+p_{n-2}N_{n-1}=q_{n-1}B. \qquad (50)$$

$$q_{n-1}M_{n-1}+q_{n-2}N_{n-1}=p_{n-1}. \qquad (51)$$

From (50) and (51) we readily deduce

$$N_{n-1}=\frac{p^2_{n-1}-q^2_{n-1}B}{p_{n-1}q_{n-2}-q_{n-1}p_{n-2}}. \qquad (52)$$

$$M_{n-1}=\frac{q_{n-1}q_{n-2}B-p_{n-1}p_{n-2}}{p_{n-1}q_{n-2}-q_{n-1}p_{n-2}}. \qquad (53)$$

It is readily seen, by reference to (18), (33), and (35), that the numerators and the common denominator of (52) and (53) always have like signs.

Consequently, N_n and M_n are positive for all values of n

Equation (47) *shows that $M_n<\sqrt{B}$; that is, M_n cannot exceed a. Equation* (43), *by transposition, becomes $y_nN_n=M_{n+1}+M_n$; which shows that N_n as well as y_n cannot exceed 2a.*

Now, since the continued fraction, which expresses $\sqrt{B}$, must extend to infinity, and since M_n as well as N_n are positive integers less than 2a, it follows that the values of M_n and N_n in $D_n=\frac{\sqrt{B}+M_n}{N_n}$ must recur in periods.

Suppose the number of terms in a period to be n, so that

$$y_{n+1}=y_1,\ M_{n+1}=M_1,\ N_{n+1}=N_1. \qquad (54)$$

If z denote the complete denominator, corresponding to the partial denominator y_n, we must have

$$z=y_n-a+\sqrt{B}. \qquad (55)$$

Hence,

$$\sqrt{B}=\frac{p_nz+p_{n-1}}{q_nz+q_{n-1}}=\frac{p_n(y_n-a)+p_n\sqrt{B}+p_{n-1}}{q_n(y_n-a)+q_n\sqrt{B}+q_{n-1}}. \qquad (56)$$

Proceeding with (56) as was done with (49), and we find

$$p_n(y_n-a)+p_{n-1}=Bq_n. \qquad (57)$$

$$q_n(y_n-a)+q_{n-1}=p_n. \qquad (58)$$

Equation (58) gives

$$\frac{p_n}{q_n}=y_n-a+\frac{q_{n-1}}{q_n}. \qquad (59)$$

Now, since $\frac{q_{n-1}}{q_n}$ cannot be a whole number, it follows that y_n-a is the greatest integral part of $\frac{p_n}{q_n}$; but, since the process begins to repeat at this point, the greatest integer of $\frac{p_n}{q_n}$ is a; therefore, $y_n-a=a$, hence, $y_n=2a$.

So that, whenever we obtain a partial denominator which is equal to twice the greatest square root of B, the process must begin to repeat.

Equation (58) gives, when $2a$ is substituted for y_n, $q_{n-1}=p_n-aq_n$, which, divided by q_n, becomes

$$\frac{q_{n-1}}{q_n}=\frac{p_n}{q_n}-a. \qquad (60)$$

Inverting both members of (16), it becomes

$$\frac{1}{q_n}=\frac{1}{y_{n-1}q_{n-1}+q_{n-2}}. \qquad (61)$$

Multiplying the numerator of the left-hand member of (61) by q_{n-1}, and dividing the denominator of the right-hand member by the same quantity, we obtain

$$\frac{q_{n-1}}{q_n}=\frac{1}{y_{n-1}+\frac{q_{n-2}}{q_{n-1}}}. \qquad (62)$$

Writing $n-1$ for n, in (62), and it gives

$$\frac{q_{n-2}}{q_{n-1}}=\frac{1}{y_{n-2}+\frac{q_{n-3}}{q_{n-2}}},$$

which substituted in (62) gives

$$\frac{q_{n-1}}{q_n}=\cfrac{1}{y_{n-1}+\cfrac{1}{y_{n-2}+\cfrac{q_{n-3}}{q_{n-2}}}}. \qquad (63)$$

Again, in (62), writing $n-2$ for n we get

$$\frac{q_{n-3}}{q_{n-2}}=\frac{1}{y_{n-3}+\frac{q_{n-4}}{q_{n-3}}},$$ which substituted in (63),

we get

$$\frac{q_{n-1}}{q_n}=\cfrac{1}{y_{n-1}+\cfrac{1}{y_{n-2}+\cfrac{1}{y_{n-3}+\cfrac{1}{\cfrac{q_{n-4}}{q_{n-3}}}}}}. \qquad (64)$$

Continuing this process we discover, that the value of $\frac{q_{n-1}}{q_n}$ is expressed by a continued fraction, less than a unit, of which the partial denominators are the same as those of the continued fractions arising from $\sqrt{B}$, taken in a reverse order.

From this, we see that if the partial denominators are symmetrical, that is, of the following form:

$$y_1, y_2, y_3, \ldots\ldots\ldots\ldots y_3, y_2, y_1, \ldots\ldots\ldots\ldots \quad (65)$$

then will $\frac{q_{n-1}}{q_n} = \frac{p_n}{q_n}$, or $q_{n-1} = p_n$, and conversely, if $q_{n-1} = p_n$, then will the partial denominators be symmetrical as given by (65.)

Now, (60) shows, that $\frac{q_{n-1}}{q_n}$ is the same as the value of $\frac{p_n}{q_n}$ in the expansion of $\sqrt{B}$, after a is subtracted.

Hence it follows, that, if we neglect the integral part $a = y$, the partial denominators of the continued fraction arising from the value of $\sqrt{B}$, will recur in symmetrical periods.

(224.) We will now extract the square root of some surds by the method of continued fractions.

From equations (43) and (44), we readily find M_{n+1}, N_{n+1}, when y_n is known. Then by condition $\frac{\sqrt{B} + M_{n+1}}{N_{n+1}} = y_{n+1} +$ &c., we readily deduce y_{n+1}. Continuing the process in this way we find in succession all the different *partial denominators*, of which the general term is y_n.

Observing the above law of derivation, we have in the case of $\sqrt{19}$, the following successive operations:

$$\frac{\sqrt{B}+M_n}{N_n}=y_n+\&c;\ N_n y_n-M_n=M_{n+1};\ \frac{B-M^2_{n+1}}{N_n}=N_{n+1}.$$

$$\frac{\sqrt{19}+0}{1}=4+\&c;\ 1\times4-0=4;\ \frac{19-4^2}{1}=3.$$

$$\frac{\sqrt{19}+4}{3}=2+\&c;\ 3\times2-4=2;\ \frac{19-2^2}{3}=5.$$

$$\frac{\sqrt{19}+2}{5}=1+\&c;\ 5\times1-2=3;\ \frac{19-3^2}{5}=2.$$

$$\frac{\sqrt{19}+3}{2}=3+\&c;\ 2\times3-3=3;\ \frac{19-3^2}{2}=5.$$

$$\frac{\sqrt{19}+3}{5}=1+\&c;\ 5\times1-3=2;\ \frac{19-2^2}{5}=3.$$

&c., &c., &c.

These operations are all so simple that most of the work can be performed mentally. Consequently, the conversion of the square root of a surd into an infinite repeating continued fraction, is a very simple matter.

If $B=28$, we shall have by proceeding as we have already done for $B=19$, the following periods of partial denominators:

$y,\ y_1,\ y_2,\ y_3,\ y_4,$
5; 3, 2, 3, 10; 3, 2, 3, 10; 3, 2, 3, 10; &c.

and our continued fraction becomes

$$\sqrt{28}=5+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{10+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{10+\&c.}}}}}}}}$$

If we compute some of the approximate fractions, by Rule, under Art. 220, we shall find

$$\begin{array}{cccccc} 5, & 3, & 2, & 3, & 10, & 3, \\ \frac{1}{0}; & \frac{5}{1}; & \frac{16}{3}; & \frac{37}{7}; & \frac{127}{24}; & \frac{1307}{247}; \text{ \&c.} \end{array}$$

$$\sqrt{28} > 5; \ \sqrt{28} < \frac{16}{3}; \ \sqrt{28} > \frac{37}{7}; \ \sqrt{28} < \frac{127}{24}; \ \sqrt{28} > \frac{1307}{247};$$

and so on for the successive values. This last value differs from the square root of 28 by a quantity less than $\frac{1}{(247)^2}$.

In the same way we find, for the square root of 31, the following partial denominators, the first term being always the integral part:

5; 1, 1, 3, 5, 3, 1, 1, 10; &c., the approximate fractions are

$$\frac{1}{0}; \frac{5}{1}; \frac{6}{1}; \frac{11}{2}; \frac{39}{7}; \frac{206}{37}; \frac{657}{118}; \frac{863}{155}; \frac{1520}{273}; \frac{16063}{2885}; \text{ \&c.}$$

The square root of 44 gives the partial denominators 6; 1, 1, 1, 2, 1, 1, 1, 12; &c., the approximate fractions are

$$\frac{1}{0}; \frac{6}{1}; \frac{7}{1}; \frac{13}{2}; \frac{20}{3}; \frac{53}{8}; \frac{73}{11}; \frac{126}{19}; \frac{199}{30}; \frac{2514}{379}; \text{ \&c.}$$

The square root of 45 gives

$$6; \ 1, 2, 2, 2, 1, 12; \text{ \&c.},$$

$$\frac{1}{0}; \frac{6}{1}; \frac{7}{1}; \frac{20}{3}; \frac{47}{7}; \frac{114}{17}; \frac{161}{24}; \frac{2046}{305}; \text{ \&c.}$$

For the square root of 53 we have

$$7; \ 3, 1, 1, 3, 14; \text{ \&c.},$$

$$\frac{1}{0}; \frac{7}{1}; \frac{22}{3}; \frac{29}{4}; \frac{51}{7}; \frac{182}{25}; \frac{2599}{357}; \text{ \&c.}$$

(225.) If we suppose s to equal the following infinite continued fraction, we have

$$s=a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+\&c.}}} \qquad (66)$$

Transposing a, and then inverting both members of (66), we have

$$\frac{1}{s-a}=2a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+\&c.}}} \qquad (67)$$

Adding a to both members of (66), it becomes

$$s+a=2a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+\&c.}}} \qquad (68)$$

Equating the left-hand members of (68) and (67), we have $s+a=\frac{1}{s-a}$; clearing of fractions, $s^2-a^2=1$; $\therefore s=\sqrt{a^2+1}$, and

$$\sqrt{a^2+1}=a+\cfrac{1}{2a+\cfrac{1}{2a+\cfrac{1}{2a+\&c.}}} \qquad (69)$$

If we make $a=1$ in (69), we find

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}} \qquad (70)$$

If we make $a=2$, we have

$$\sqrt{5}=2+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\&c.}}} \qquad (71)$$

Again, suppose

$$s=a+\cfrac{1}{2+\cfrac{1}{2a+\cfrac{1}{2+\cfrac{1}{2a+\&c.}}}} \qquad (72)$$

Transposing a and inverting both members, we have

$$\frac{1}{s-a}=2+\cfrac{1}{2a+\cfrac{1}{2+\cfrac{1}{2a+\cfrac{1}{2+\&c.}}}} \qquad (73)$$

Transposing 2 and inverting, we have

$$\frac{s-a}{1-2s+2a}=2a+\cfrac{1}{2+\cfrac{1}{2a+\cfrac{1}{2+\&c.}}} \qquad (74)$$

Adding a to both members of (72), and it becomes

$$s+a=2a+\cfrac{1}{2+\cfrac{1}{2a+\cfrac{1}{2+\&c.}}} \qquad (75)$$

Equating the left-hand members of (75) and (74), we have

$$s+a=\frac{s-a}{1-2s+2a}. \qquad (76)$$

Clearing this of fractions, and reducing, we have

$$s=\sqrt{a^2+a}\ \therefore \sqrt{a^2+a}=a+\cfrac{1}{2+\cfrac{1}{2a+\cfrac{1}{2+\cfrac{1}{2a+\&c.}}}} \quad (77)$$

If we take $a=2$, in (77), we have

$$\sqrt{6}=2+\cfrac{1}{2+\cfrac{1}{4+\cfrac{1}{2+\cfrac{1}{4+\&c.}}}}$$

Making $a=13$, we have

$$\sqrt{182}=13+\cfrac{1}{2+\cfrac{1}{26+\cfrac{1}{2+\cfrac{1}{26+\&c.}}}}$$

(226.) A continued fraction, and consequently any common fraction, can be converted into a series as follows:

Equation (18) gives

$$\frac{p_2}{q_2}-\frac{p_1}{q_1}=\frac{1}{q_1q_2}.$$

$$\frac{p_3}{q_3}-\frac{p_2}{q_2}=\frac{-1}{q_2q_3}.$$

$$\frac{p_4}{q_4}-\frac{p_3}{q_3}=\frac{1}{q_3q_4}.$$

$$\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots$$

$$\frac{p_n}{q_n}-\frac{p_{n-1}}{q_{n-1}}=\frac{(-1)^n}{q_{n-1}q_n}.$$

Hence, by addition,

$$\frac{p_n}{q_n}=\frac{p_1}{q_1}+\frac{1}{q_1q_2}-\frac{1}{q_2q_3}+\frac{1}{q_3q_4}-\ldots\ldots+\frac{(-1)^n}{q_{n-1}q_n}. \quad \text{(A)}$$

The terms of this series continually decrease, and are alternately positive and negative; consequently the error committed by taking n terms of the series is less than the $(n+1)$th term.

CHAPTER IX.

LOGARITHMS.

(227.) *Logarithms* are numbers, by the aid of which many arithmetical operations are greatly simplified.

In the following relations:

$$\left.\begin{array}{ll} a^x = b, & (1) \\ a^y = c, & (2) \\ a^z = d, & (3) \\ \&c., & \&c. \end{array}\right\} \quad (A)$$

x, y and z are respectively the logarithms of b, c, and d.

(228.) The assumed root a is called the base of the system of logarithms.

(229.) If in (1), of equations (A), we make $x = 0$, we shall have $a^0 = b = 1$, for all values of a, *therefore the logarithm of* 1 *is always* 0.

(230.) If in (1), we suppose the base to be negative, we shall have $(-a)^x = b$. If b is positive, then x must be even, if b is negative, then x must be odd; hence we can not represent all values of b by the expression $(-a)^x$. *Therefore the base of every system of logarithms must be positive.*

(231.) If in (1), we suppose b to be negative, we shall have $a^x = -b$. Now, since a is always positive, the expression a^x is positive for all values of x either positive or negative.

Therefore, the logarithm of a negative quantity is impossible.

(232.) Each different base must produce a different system of logarithms; the logarithms in common use have 10 for their base.

So that we have

$$10^0 = 1;\ 10^1 = 10;\ 10^2 = 100;\ 10^3 = 1000;\ \&c.$$

Hence, we have

$$\begin{array}{ll} \log.\ 1 = 0, & \log.\ \frac{1}{10} = -1, \\ \log.\ 10 = 1, & \log.\ \frac{1}{100} = -2, \\ \log.\ 100 = 2, & \log.\ \frac{1}{1000} = -3, \\ \log.\ 1000 = 3, & \log.\ \frac{1}{10000} = -4, \\ \log.\ 10000 = 4, & \&c., \\ \&c. & \end{array}$$

(233.) If we take the product of equations (1) and (2) of group (A), we shall have

$$a^{x+y} = bc, \qquad (4)$$

from which we discover that, *the logarithm of the product of two quantities is equal to the sum of their logarithms.*

And in general, the logarithm of a number consisting of any number of factors is equal to the sum of the logarithms of all its factors.

(234.) It also follows from the above, *that n times the logarithm of any number is equal to the logarithm of its nth power.*

(235.) If we divide equation (1) by (2), of group (A), we shall find

$$a^{x-y}=\frac{b}{c}, \qquad (5)$$

from which we see *that, the difference of the logarithms of any two quantities is equal to the logarithm of their quotient.*

(236.) We have just shown that the logarithm of a number raised to the nth power, is equal to n times the logarithm of the number. *Conversely, the logarithm of the* nth *root of a number, is equal to the* nth *part of the logarithm of the number.*

(237.) We will now show how the numerical values of logarithms may be found.

If x is the logarithm of N for the base a we shall have this condition :

$$a^x=N. \qquad (6)$$

If we assume

$$\left.\begin{aligned} a&=1+m, \\ N&=1+n, \end{aligned}\right\} \qquad (7)$$

we shall have

$$(1+m)^x=1+n. \qquad (8)$$

Involving both members of this to the yth power, we shall have

$$(1+m)^{xy}=(1+n)^y. \qquad (9)$$

By the *Binomial Theorem*, we find

$$(1+m)^{xy}=$$

$$1+xym+\frac{xy(xy-1)}{1.2}.m^2+\frac{xy(xy-1)(xy-2)}{1.2.3}.m^3+\text{\&c.},$$

$$(1+n)^y=1+yn+\frac{y(y-1)}{1.2}.n^2+\frac{y(y-1)(y-2)}{1.2.3}.n^3+\text{\&c.}$$

Equating these expanded values, rejecting the units of both expressions, we have, after dividing through by y,

$$x\left\{m+\frac{xy-1}{2}.\,m^2+\frac{(xy-1)\,(xy-2)}{2.3}.\,m^3+\ \&c.\right\}=$$
$$n+\frac{y-1}{2}.\,n^2+\frac{(y-1)\,(y-2)}{2.3}.\,n^3+\ \&c.$$

This must be true for all values of y.

When $y=0$, it becomes

$$x\{m-\tfrac{1}{2}m^2+\tfrac{1}{3}m^3-\tfrac{1}{4}m^4+\&c.\}=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\&c.$$

Hence,

$$x=\log.\mathcal{N}=\log.(1+n)=\frac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\&c.}{m-\frac{1}{2}m^2+\frac{1}{3}m^3-\frac{1}{4}m^4+\&c.}\quad(10)$$

Re-substituting $a-1$ for m, and we have

$$\left.\begin{array}{l}\log.(1+n)=\\ \dfrac{n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\ \&c.}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.}\end{array}\right\}\quad(11)$$

If we assume

$$M=\frac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\&c.}$$

we shall have

$$\log.(1+n)=M\,\{n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\ \&c.\}.\quad(B)$$

If the base be so chosen as to render $M=1$, then formula (B) will become

$$\log.(1+n)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\ \&c.\quad(C)$$

(238.) The logarithms obtained by formula (C) are called *hyperbolic* or *Napierean*, whilst the common logarithms given by formula (B), are called *Briggean*.

Lord Napier, or Neper, is supposed to have first constructed logarithms. The logarithms in common use, were first calculated by Mr. Briggs.

(239.) We shall hereafter denote the Napierean logarithms by the abbreviation $\mathcal{N}$ log., whilst the common or

Briggean logarithms will be represented simply by log. Hence formula (C) will become

$$\mathcal{N}\log.(1+n)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\&c. \quad (C')$$

(240.) By comparing formulas (B) and (C′) we discover this relation

$$M\times\mathcal{N}\log.(1+n)=\log.(1+n). \qquad (12)$$

Therefore,

$$M=\frac{\log.(1+n)}{\mathcal{N}\log.(1+n)}. \qquad (D)$$

$M=\dfrac{1}{(a-1)-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\&c.}$, is called the *modulus* of the system of logarithms whose base is a.

From (12) we see that, *the logarithms of any particular system is equal to the Napierean logarithm multiplied by the modulus of that particular system.*

(241.) We will now proceed to calculate some Napierean logarithms.

Resuming formula (C′), which is

$$\mathcal{N}\log.(1+n)=n-\tfrac{1}{2}n^2+\tfrac{1}{3}n^3-\tfrac{1}{4}n^4+\tfrac{1}{5}n^5-\&c.$$

we have, when n is made negative,

$$\mathcal{N}\log.(1-n)=-n-\tfrac{1}{2}n^2-\tfrac{1}{3}n^3-\tfrac{1}{4}n^4-\&c. \quad (1)$$

Subtracting (1) from (C′), we have

$$\left.\begin{aligned}\mathcal{N}\log.(1+n)-\mathcal{N}\log.(1-n)&=\mathcal{N}\log.\frac{1+n}{1-n}\\ &=2(n+\tfrac{1}{3}n^3+\tfrac{1}{5}n^5+\tfrac{1}{7}n^7+\tfrac{1}{9}n^9+\&c.)\end{aligned}\right\} \quad (2)$$

If we assume $n=\dfrac{1}{2p+1}$, we shall find $\dfrac{1+n}{1-n}=\dfrac{p+1}{p}$, and (2) will become

$$\mathcal{N}\log.\frac{p+1}{p}=2\left\{\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+\&c.\right\} \quad (3)$$

Or, which is the same thing,

$\mathcal{N}\log.\,(p+1)=$

$$\mathcal{N}\log.\,p+2\left\{\frac{1}{2p+1}+\frac{1}{3(2p+1)^3}+\frac{1}{5(2p+1)^5}+\text{\&c.}\right\} \quad \text{(E)}$$

If we take $p=1$, formula (E) becomes

$$\mathcal{N}\log.\,2=2\left\{\frac{1}{3}+\frac{1}{3.3^3}+\frac{1}{5.3^5}+\frac{1}{7.3^7}+\text{\&c.}\right\}$$

$$\begin{array}{r|l}
3 & 2 \\
3^2=9 & 0.66666666\div\ \ 1=0.66666666 \\
9 & 0.07407407\div\ \ 3=0.02469136 \\
9 & 0.00823045\div\ \ 5=0.00164609 \\
9 & 0.00091449\div\ \ 7=0.00013064 \\
9 & 0.00010161\div\ \ 9=0.00001129 \\
9 & 0.00001129\div 11=0.00000103 \\
9 & 0.00000125\div 13=0.00000010 \\
9 & 0.00000014\div 15=0.00000001 \\
 & 0.00000001 \qquad\qquad \overline{} \\
\end{array}$$

$$0.69314718=\mathcal{N}\log.\,2.$$

Take $p=4$, in formula (E), and we get

$$\mathcal{N}\log.\,5=\mathcal{N}\log.\,4+2\left\{\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\text{\&c.}\right\}$$

But $\qquad \mathcal{N}\log.\,5=\mathcal{N}\log.\,10-\mathcal{N}\log.\,2;$

also, $\qquad \mathcal{N}\log.\,4=2\,\mathcal{N}\log.\,2.$

Hence, substituting these values of $\mathcal{N}\log.\,5$ and $\mathcal{N}\log.\,4$, in the above expression, and we get, after transposing,

$$\mathcal{N}\log.\,10=3\,\mathcal{N}\log.\,2+2\left\{\frac{1}{9}+\frac{1}{3.9^3}+\frac{1}{5.9^5}+\frac{1}{7.9^7}+\text{\&c.}\right\}$$

Executing the calculation, for the sum of the series, as in the above example, omitting the ciphers on the left, we obtain the following:

$$
\begin{array}{r|l r}
9 & 2 & \\
9 & 0.22222222 \div 1 = & 0.22222222 \\
9 & \quad 2469136 & \\
9 & \quad\; 274348 \div 3 = & 91449 \\
9 & \qquad 30483 & \\
9 & \qquad\; 3387 \div 5 = & 677 \\
9 & \qquad\quad 376 & \\
9 & \qquad\quad\; 42 \div 7 = & 6 \\
 & \qquad\qquad 5 & \overline{} \\
\end{array}
$$

$$
\begin{aligned}
& 0.22314354 = \text{sum of series.} \\
3\ \mathcal{N}\text{log.}2 = {} & 2.07944154 \\
& \overline{2.30258508} = \mathcal{N}\text{log. } 10.
\end{aligned}
$$

We are now prepared to find the modulus of the Briggean system. Since the base of the Briggean system is 10, and the logarithm of the base of any system is 1, we have log. $10=1$; formula (D) shows, *that the common logarithm of any number divided by the Napierean logarithm is equal to the modulus of the common system.*

Hence,

$$M = \frac{\text{log. } 10}{\mathcal{N}\text{log. } 10} = \frac{1}{2.30258508} = 0.43429448.$$

This value, when carried to 35 decimal places, is

$$M = 0.43429448190325182765112891891660508.$$

We will now proceed to calculate common logarithms.

Since all numbers are either primes, or composed of a certain number of *prime* factors, and since the logarithm of any number is equal to the sum of the logarithms of all its factors, it follows that it will be necessary only to calculate the logarithms of prime numbers.

By equation (12), Art. 240, we see that the Napierean logarithm multiplied by M, gives the common logarithm.

Hence,

$$\log. 2 = N \log. 2 \times M = \\ 0.69314718 \times 0.43429448 = 0.30103000.$$

The logarithm of 10 we know to be 1, therefore the

$$\log. 5 = \log. \frac{10}{2} = 1 - \log. 2 = 0.69897000.$$

Formula (E), when adapted to common logarithms, becomes

$$\left.\begin{array}{l}\log. (p+1) = \\ \log. p + 2M\left\{\dfrac{1}{2p+1} + \dfrac{1}{3(2p+1)^3} + \dfrac{1}{5(2p+1)^5} + \&c.\right.\end{array}\right\}$$

or,

$$\left.\begin{array}{l}\log. (p+1) = \log. p + \\ 0.86858896\left\{\dfrac{1}{2p+1} + \dfrac{1}{3(2p+1)^3} + \dfrac{1}{5(2p+1)^5} + \&c.\right\}\end{array}\right\} \text{(F)}$$

Take $p = 2$ in (F), and we get

$$\log. 3 = \log. 2 + 0.86858896\left\{\frac{1}{5} + \frac{1}{3.5^3} + \frac{1}{5.5^5} + \frac{1}{7.5^7} + \&c.\right\}$$

$$\begin{array}{r|rr}
5 & 0.86858896 & \\
5^2 = 25 & 0.17371779 \div 1 = & 0.17371779 \\
25 & 694871 \div 3 = & 231624 \\
25 & 27795 \div 5 = & 5559 \\
25 & 1112 \div 7 = & 159 \\
25 & 44 \div 9 = & 5 \\
 & 2 & \overline{} \\
 & & 0.17609126 = \text{sum of series} \\
 & \log. 2 = & 0.30103000 \\
 & & \overline{0.47712126} = \log. 3.
\end{array}$$

If, in (F), we make $p=49$, we get

log. 50 =

$$\text{log. } 49+0.86858896\left\{\frac{1}{99}+\frac{1}{3.(99)^3}+\frac{1}{5.(99)^5}+\&c.\right\}$$

And since

log. 50 = log. 10 + log. 5, and log. 49 = 2 log. 7,

we have by substitution and transposition,

2 log. 7 =

$$\text{log. } 10+\text{log. } 5-0.86858896\left\{\frac{1}{99}+\frac{1}{3.(99)^3}+\frac{1}{5.(99)^5}+\&c\right\}$$

Calculating the series, we find

$$\begin{array}{r|l}
99 & 0.86858896 \\
(99)^2=9801 & \quad 877362\div1=0.00877362 \\
 & \qquad\quad 89\div3= \qquad\qquad 29 \\
\hline
 & \qquad\qquad\qquad 0.00877391=\text{sum of series.}
\end{array}$$

$$\begin{array}{r}
\text{log. } 5=0.69897000 \\
\text{log. } 10=1. \qquad\quad \\
\hline
1.69897000 \\
0.00877391 \\
\hline
1.69019609=2\text{ log. }7, \\
0.84509804=\text{log. }7.
\end{array}$$

We might have calculated the log. 7 by substituting 6 for p in (F), but the operation would have been more lengthy than the above.

The next prime in order is 11; to find its log. we make, in equation (F), $p=99$, observing that log. $100=2$, also, that

$$\log.99 = \log.\ 9 + \log.\ 11 = 2\log.\ 3 + \log.\ 11,$$

we thus obtain

$$2 = 2\log.3 + \log.11 + 0.86858896\left\{\frac{1}{199} + \frac{1}{3(199)^3} + \&c.\right\}$$

Or, by transposing, it becomes

$$\log.11 = 2 - 2\log.3 - 0.86858896\left\{\frac{1}{199} + \frac{1}{3(199)^3} + \&c.\right\}$$

$$\begin{array}{r|l}
199 & 0.86858896 \\
39601 & \quad 436477 \div 1 = 0.00436477 \\
 & \qquad\quad 11 \div 3 = \qquad\qquad 4 \\
\end{array}$$

$$\begin{array}{rl}
 & \overline{0.00436481} = \text{sum of series.} \\
2\log.\ 3 = & 0.95424252 \\
 & \overline{0.95860733}
\end{array}$$

$$\begin{array}{r}
2.00000000 \\
0.95860733 \\
\hline
1.04139267 = \log.\ 11.
\end{array}$$

To find the log. of the next prime 13, we assume $p = 1000$ in equation (F), and obtain

$$\log.\ 1001 =$$
$$\log.\ 1000 + 0.86858896\left\{\frac{1}{2001} + \frac{1}{3(2001)^3} + \&c.\right\}$$

Now, since

$$1001 = 7 \times 11 \times 13, \log.\ 1001 = \log.\ 7 + \log.\ 11 + \log.\ 13$$

Hence,

$$\log.\ 13 =$$
$$3 - \log.7 - \log.11 + 0.86858896\left\{\frac{1}{2001} + \frac{1}{3(2001)^3} + \&c.\right\}$$

$$2001 \,|\, 0.86858896$$

$$43407 = \text{sum of series.}$$

$$\log.\ 7 = 0.84509804$$

$$\log.\ 11 = 1.04139267$$

$$1.88649071$$

$$3.00043407$$
$$1.88649071$$
$$1.11394336 = \log.\ 13.$$

We might proceed in this way until we should have calculated the logarithms of all the prime numbers within the limits of the tables.

If, in formula (F), we substitute q^2-1 for p, it will become

$$\log.\ q^2 = \log.\ (q^2-1) + 2M\left\{\frac{1}{2q^2-1} + \frac{1}{3(2q^2-1)^3} + \&c.\right\}$$

Now, since $\log.\ q^2 = 2\log.\ q$,

and $\quad \log.\ (q^2-1) = \log.\ (q+1) + \log.\ (q-1)$,

we have

$$\log.\ (q+1) = 2\log.\ q$$
$$-\log.(q-1) - 0.86858896\left\{\frac{1}{2q^2-1} + \frac{1}{3(2q^2-1)^3} + \&c.\right\}$$

When $q > 13$, we have this very simple formula:

$$\log.(q+1) = 2\log.\ q - \log.(q-1) - \frac{0.86858896}{2q^2-1}. \quad \text{(G)}$$

This formula will be true to 8 places of decimals.

Having already obtained the logarithms of all numbers as far as 13, we may now make use of formula (G) for all numbers exceeding 13, and thus shorten the labor.

(242.) We have already (Art. 237) said, that the base a of the Napierean system of logarithms satisfies the following equation:

$$(a-1)-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\&c.=1. \quad (1)$$

From example 3, page 260, we see that if we have

$$(y-1)-\frac{(y-1)^2}{2}+\frac{(y-1)^3}{3}+\frac{(y-1)^4}{4}+\&c.=x, \quad (2)$$

then will

$$y=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\&c. \quad (3)$$

Equation (2) will agree with (1) when $y=a$, and $x=1$. Making these changes in (3), we find

$$a=1+1+\frac{1}{2}+\frac{1}{2.3}+\frac{1}{2.3.4}+\frac{1}{2.3.4.5}+\&c. \quad (4)$$

This series may be summed as follows:

	1
2	1
3	0.5
4	0.16666666
5	4166666
6	833333
7	138888
8	19841
9	2480
10	276
11	28
	2

2.71828180 = base of Napierean logarithms.

This value, when extended to 35 decimals, is found to be

$e=2.71828182845904523536028747135266249.$

EXPONENTIAL THEOREM.

(243.) This theorem makes known the law of the development of a^x according to the ascending powers of x.

To determine this law, we will assume

$$a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \&c., \qquad (1)$$

both members of which become 1, when $x=0$.

Changing x into y, in (1), and we have

$$a^y = 1 + Ay + By^2 + Cy^3 + Dy^4 + \&c. \qquad (2)$$

Subtracting (2) from (1), and actually dividing the right-hand member by $x-y$, we obtain

$$\frac{a^x - a^y}{x-y} = \left.\begin{array}{l} A + B(x+y) + C(x^2+xy+y^2) \\ \quad + D(x^3+x^2y+xy^2+y^3) + \&c. \end{array}\right\} \qquad (3)$$

Writing $x-y$ for x, in (1), and it becomes

$$a^{x-y} = 1 + A(x-y) + B(x-y)^2 + C(x-y)^3 + \&c. \qquad (4)$$

Transposing the 1, and multiplying by a^y, we get

$$a^y(a^{x-y}-1) = a^y[A(x-y)+B(x-y)^2+C(x-y)^3+\&c.] \qquad (5)$$

Dividing (5) by $x-y$, after replacing its left-hand member by its equivalent value $a^x - a^y$, we find

$$\frac{a^x-a^y}{x-y} = a^y[A+B(x-y)+C(x-y)^2+D(x-y)^3+\&c.] \qquad (6)$$

Equating the right-hand members of (3) and (6), we have

$$\left\{\begin{array}{l} \left.\begin{array}{l} A+B(x+y)+C(x^2+xy+y^2) \\ \quad + D(x^3+x^2y+xy^2+y^3)+\&c. \end{array}\right\} \\ = a^y[A+B(x-y)+C(x-y)^2+D(x-y)^3+\&c.] \end{array}\right\} \qquad (7)$$

This is true for all values of x and y.

When $y=x$, it becomes

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. = a^x \,.\, A. \qquad (8)$$

For a^x, substituting its value, equation (1), we find

$$\left.\begin{array}{l} A + 2Bx + 3Cx^2 + 4Dx^3 + \&c. \\ \quad = A + A^2x + ABx^2 + ACx^3 + \&c. \end{array}\right\} \qquad (9)$$

Equating the coefficients of the like powers of x, (Art 183), we find

$$\left.\begin{aligned} A &= A,\\ 2B &= A^2,\\ 3C &= AB,\\ 4D &= AC,\\ &\text{\&c., \&c.}\end{aligned}\right\}\text{ Therefore, }\left\{\begin{aligned} A &= A,\\ B &= \frac{A^2}{2},\\ C &= \frac{A^3}{2.3},\\ D &= \frac{A^4}{2.3.4},\\ &\text{\&c. \&c.}\end{aligned}\right.$$

Hence, (1) becomes

$$a^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4} + \text{\&c.} \qquad (10)$$

It now remains to find the value of A.

For this purpose, put $1+b=a$, and we have $a^x = (1+b)^x$, which, by the Binomial Theorem, becomes

$$\left.\begin{aligned}&(1+b)^x =\\ &1 + \frac{xb}{1} + \frac{x(x-1)b^2}{1.2} + \frac{x(x-1)(x-2)b^3}{1.2.3} + \text{\&c.}\end{aligned}\right\} \qquad (11)$$

Performing the multiplications indicated, we find the coefficient of the first power of x to be

$$\frac{b}{1} - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \text{\&c.},$$

or, re-substituting $a-1$ for b, it becomes

$$(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}$$

Therefore,

$$A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.} \quad (12)$$

If in formula (C′), we put $a-1$ for n, we shall find

$$N\log.a = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}$$

Hence,

$$\mathcal{A}=\mathcal{N}\log.\ a. \qquad (13)$$

This value of $\mathcal{A}$ substituted in (10), gives

$$a^x=1+\mathcal{N}\log.\ a.\ x+\frac{(\mathcal{N}\log.\ a)^2.x^2}{2}+\frac{(\mathcal{N}\log.\ a)^3.x^3}{2.3}+ \qquad (A)$$

When

$$a=e=2.7182818\ \&c.,$$

then

$$\mathcal{N}\log.\ a=\mathcal{N}\log.\ e=1,$$

and (A) becomes

$$e^x=1+x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{4.3.4}+\ \&c. \qquad (B)$$

APPLICATION OF LOGARITHMS.

(244.) By the aid of a table of logarithms, we can easily perform the following operations :

1. Find the value of $\frac{3.75\times1.06}{365}$ by logarithms.

Recollecting (Art. 233) that the logarithm of the product of several factors is equal to the sum of their respective logarithms ; and (Art. 235) the logarithm of the quotient of one quantity divided by another is equal to the logarithm of the dividend diminished by the logarithm of the divisor, we find for the logarithm of our expression

$$\log.\ \frac{3.75\times1.06}{365}=\log.\ 3.75+\log.\ 1.06-\log.\ 365.$$

By the tables we have

$$\begin{aligned}\log.\ 3.75&=0.5740313\\ \log.\ 1.06&=\underline{0.0253059}\\ &0.5993372\\ \log.\ 365&=\underline{2.5622929}\\ \log.\ 0.01089&=\bar{2}.0370443.\end{aligned}$$

Therefore, the above expression is nearly equal to 0.01089.

2. Find the 11th root of 11, that is, the value of the $\sqrt[11]{11}$.

Taking the logarithm of this expression, we find

$$\log. \sqrt[11]{11} = \tfrac{1}{11} \text{ of } \log. 11 = \tfrac{1}{11} \text{ of } 1.0413927 = 0.0946721$$
$$= \log. 1.24357 \text{ \&c.}$$

Therefore, $\sqrt[11]{11} = 1.24357.$

3. What is the value of $\dfrac{8^5 \times \sqrt[3]{7}}{\sqrt[3]{6}}$?

$$5 \times \log. 8 + \tfrac{1}{3} \log. 7 - \tfrac{1}{3} \log. 6 = 4.51545 + 0.2816993$$
$$-0.1556302 = 4.6415191 = \log. 43794.53.$$

Therefore, our expression is equivalent to 43804.53.

(245.) The above examples will show the great advantage of logarithms in abridging arithmetical labor. In the higher parts of analysis, the use of logarithms is indispensable. It would not be difficult to propose questions, which by logarithms might be wrought in a few moments, but if wrought by arithmetical rules, would require years. The following example will illustrate the above remark.

How many figures will be required to express 9^{9^9}?

The exponent of the above expression is

$$9^9 = 387420489 \quad \therefore \; 9^{9^9} = 9^{387420489}.$$

Putting it into logarithms, we have

$$\log. 9^{387420489} = 387420489 \times \log. 9 =$$
$$387420489 \times 0.954242509439 = 369693099.63 \text{ \&c.}$$

Hence, the number answering to this logarithm must consist of 369693100 figures. This number, if printed, would fill upwards of 256 volumes of 400 pages each, allowing 60 lines to a page, and 60 figures to a line.

EXPONENTIAL EQUATIONS.

(246.) An exponential equation is one where the unknown quantity enters as an exponent.

Thus, $a^x = b$; $x^x = c$; &c.,

are exponential equations.

(247.) When the equation is of the form $a^x = b$, we find, by taking the logarithm of both members, $x \times \log. a = \log. b$.

Therefore, $$x = \frac{\log. b}{\log. a}.$$

(248.) When the exponential is of this form $x^x = c$, we must find the value of x by the following double position

RULE.

Find by trial two numbers as near the value of x as possible, and substitute them successively for x; then, as the difference of the results is to the difference of the two assumed numbers, so is the difference of the true result, and either of the former, to the difference of the true number and the supposed one belonging to the result last used; this difference, therefore, being added to the supposed number, or subtracted from it, according as it is too little or too great, will give the true value nearly.

And if this near value be substituted for x, as also the nearest of the first assumed numbers, unless a number still nearer be found, and the above operations be repeated, we shall obtain a still nearer value of x; and in this way we may continually approximate to the true value of x.

EXAMPLES.

1. Given $x^x = 100$, to find an approximate value of x.

The above equation, when put into logarithms, becomes

$$x \times \log. x = \log. 100 = 2. \qquad (1)$$

By a few trials we find the value of x to fall between 3 and 4. If we substitute, in succession, 3 and 4 in (1), we shall find

$$3 \times \log. 3 = 1.4313639$$
$$4 \times \log. 4 = 2.4082400$$
$$0.9768761 = \text{diff. of results.}$$

$$0.9768761 : 1 :: 0.4082400 : 0.418.$$

Hence, $\quad 4 - 0.418 = 3.582 = x$ nearly.

Upon trial, this value is found to be rather too small, whilst 3.6 is rather too great; therefore, substituting each of these in succession in (1), we find,

$$3.582 \times \log. 3.582 = 1.9848779$$
$$3.6 \times \log. 3.6 = 2.0026890$$
$$0.0178111 = \text{diff. of results.}$$

$$0.0178111 : 0.018 :: 0.0026890 : 0.002717.$$

Hence,

$$3.6 - 0.002717 = 3.597283 = x \text{ nearly.}$$

2. Given $x^x = 5$, to find an approximate value of x.

Ans. $x = 2.1293$.

3. Given $x^x = 2000$, to find an approximate value of x

Ans. $x = 4.8278$.

4. Given $x^{x^x} = 100$, to find an approximate value of x.

Ans. $x = 2.2127$.

COMPOUND INTEREST AND ANNUITIES BY LOGARITHMS.

(249.) *Interest* is money paid by the borrower for the use of the money borrowed.

It is estimated at a certain *rate per cent. per annum*; that is, a certain number of dollars for the use of $100 for one year.

The sum upon which the interest is computed is called the *principal*.

The principal, when increased by the interest, is called the *amount*.

When the interest of a given principal is paid at the end of each year, it is called *simple* interest; but when the interest due, at the end of each year, goes to increase the principal, it is called *compound* interest.

The *present worth*, at compound interest, of a given debt, due at some future time, is such a sum as, being put out at compound interest, will, in the given time, amount to the debt.

An annuity is a fixed sum which is paid periodically, for a certain length of time.

(250.) In our calculations we shall use the following notation:

$p =$ the principal.
$r =$ the interest of \$1 for one year.
$R = \$1 + r =$ the amount of \$1 for one year.
$a =$ the amount of the given principal.
$A =$ an annuity.
$a' =$ the amount of a given annuity.
$P =$ the present worth of a given annuity.
$n =$ the time in years.

Since $\$1 + r = R$ is the amount of \$1 for one year, it follows, that the amount of a given principal, p, will in the same time be pR, and this being considered as a new principal, will in the next year amount to $pR \times R = pR^2$, which,

in turn, will the next year amount to $pR^2 \times R = pR^3$; and so on.

Hence,

$$\left.\begin{array}{l} pR = \text{amount for 1 year.} \\ pR^2 = \text{amount for 2 years.} \\ pR^3 = \text{amount for 3 years.} \\ pR^4 = \text{amount for 4 years.} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ pR^n = \text{amount for } n \text{ years.} \end{array}\right\}$$

Therefore, we have this relation,

$$a = pR^n,$$

which, in logarithms, becomes

$$\log. a = \log. p + n \log. R. \qquad (1)$$

(251.) When an annuity is left unpaid for n years, it is obvious that the annuity due at the end of the first year, must be on interest $n-1$ years, and must therefore amount to AR^{n-1}; the annuity due at the end of the second year will be on interest $n-2$ years and will therefore amount to AR^{n-2}, and so on, hence, the amount of the annuity at the end of n years will be

$$A(R^{n-1} + R^{n-2} + \ldots\ldots\ldots\ldots R + 1).$$

The geometrical progression within the parenthesis being summed, we have, after substituting r for $R-1$,

$$a' = A\left(\frac{R^n - 1}{r}\right). \qquad (2)$$

We have said that the present worth of a debt is such a sum as being put out at interest, will, in the given time, amount to the debt, hence we have

$$PR^n = A\left(\frac{R^n - 1}{r}\right), \qquad (2')$$

from which we find

$$P=\frac{A\left(1-\frac{1}{R^n}\right)}{r}=\frac{a'}{R^n}. \qquad (3)$$

When the annuity is continued *forever*, the value of n becomes infinite, making this substitution in (3), we find

$$P=\frac{A}{r}. \qquad (4)$$

The amount of \$1 at compound interest for n years at r per cent., is

$$(1+r)^n. \qquad (5)$$

The amount of \$1 at simple interest for n years at r per cent., is

$$1+nr. \qquad (6)$$

Expression (5), when expanded by the binomial theorem, becomes.

$$(1+r)^n=1+nr+\frac{n(n-1)}{2}r^2+\frac{n(n-1)(n-2)}{2.3}r^3+\&c.$$

When $n=1$, this expression becomes

$$1+nr.$$

When $n>1$, this expression is $>1+nr$.

When $n<1$, it is $<1+nr$.

Hence, the compound interest computed by formula (1), is equal to the simple interest when the time is one year; it is greater than the simple interest, when the time is greater than one year, and less than the simple interest, when the time is less than one year.

EXAMPLES.

1. How much will \$875 amount to in 12 years, at 6 per cent, compound interest?

In this example, we have

$$p = 875\ ;\ n = 12\ ;\ R = 1.06\ ;$$

and we are required to find a.

Substituting these values in equation (1), we have.

$$\log.\ a = \log.\ 875 + 12 \log.\ 1.06.$$

By actually consulting a table of logarithms, we find

$$\begin{aligned} \log.\ 875 &= 2.9420081 \\ 12 \log.\ 1.06 &= 0.3036708 \\ \hline \log.\ a &= 3.2456789. \end{aligned}$$

Therefore, $a = \$1760.672.$

2. What principle will, in 10 years, at 5 per cent., amount to \$1000?

By transposition, equation (1) becomes

$$\log.\ p = \log.\ a - n \log.\ R.$$

Substituting for a, n and R their given values, we have

$$\log.\ p = \log.\ 1000 - 10 \log.\ 1.05,$$

$$\therefore \log.\ p = 3 - 0.2118930 = 2.7881070.$$

And, $p = \$613.913.$

3. At what rate per cent, will \$100 in 16 years amount to \$160?

Fquation (1) gives

$$\log.\ R = \frac{\log.\ a - \log.\ p}{n},$$

which, in this example, becomes

$$\log. R = \frac{2.2041200 - 2}{16} = 0.0127575.$$

$$\therefore R = 1.02981.$$

Therefore, the per cent. is 2.981, or nearly 3 per cent.

4. In how many years will \$460 at 7 per cent., amount to \$1000?

Again, equation (1) gives

$$n = \frac{\log. a - \log. p}{\log. R},$$

which, in this example, becomes

$$n = \frac{3 - 2.6627578}{0.0293838} = 11.477 \text{ years, nearly.}$$

5. What is the amount of an annuity of \$200, which has remained unpaid 14 years, at 6 per cent., compound interest?

Equation (2), when put into logarithms, becomes

$$\log. a' = \log. A + \log. (R^n - 1) - \log. r.$$

In the present example

$$r = 0.06;\ R = 1.06;\ A = 200;\ n = 14.$$

$$\log. R^n = n \log. R = 0.3542826.$$

$$\therefore R^n = 2.2609 \text{ and } R^n - 1 = 1.2609.$$

Hence,

$$\log. a' = 2.3010300 + 0.1006806 - \bar{2}.7781513,$$

and

$$\log. a' = 3.6235593.$$

Therefore,

$$a' = \$4203 \text{ nearly.}$$

6. What is the present worth of the above annuity?

Equation (3) gives

$$\log. P = \log. a' - n \log. R.$$

In this particular case, we have

$$\log.\ P = 3.6235593 - 0.3542826 = 3.2692767.$$

and $$P = \$1858.988.$$

7. What is the present worth of an annuity of \$100, to continue forever, at 7 per cent?

By equation (4), which is $P = \frac{A}{r}$, we find

$$P = \frac{\$100}{0.07} = \$1428.571.$$

8. A debt, due at the present time, amounting to \$1200, is to be discharged in seven yearly and equal payments. What is the amount of one of these payments, if the interest is calculated at 4 per cent?

In this example, we have given the present worth of an annuity, the time of its continuance and the rate of interest, to find the annuity.

Equation (2′), by a slight reduction, becomes

$$A = \frac{PrR^n}{R^n - 1},$$

which, in logarithms, is

$$\log.\ A = \log.\ P + \log.\ r + n \log.\ R - \log.\ (R^n - 1).$$

If we take

$$P = \$1200\ ;\ r = 0.04\ ;\ R = 1.04\ ;\ n = 7,$$

we shall find

$$A = \$199.931.$$

9. In what time will a given principal, at compound interest, amount to m times the principle?

Under example 4, we have the formula

$$n = \frac{\log.\ a - \log.\ p}{\log.\ R},$$

To make this agree with the present case, we must, for a, write mp, by which means it becomes

$$n=\frac{\log.\ m}{\log.\ R}.$$

(252.) When the interest, instead of being added to the principal at the end of each year, is added at any other regular period, as half yearly, quarterly, &c., n must be considered as standing for the number of those periods, and r will be the interest for one of those periods.

10. What is the amount of \$100, for three years, at 6 per cent. per annum, when the interest is added at the end of every 6 months?

Equation (1), when adapted to the present example, becomes

$$\log.\ a=\log.\ 100+6\ \log.\ 1.03,$$

from which we find

$$a=\$119.405$$

11. If the interest of \$1, for the xth part of a year, is $\frac{r}{x}$, what will be the amount of \$1 for n years, when $x=\infty$?

The formula for the amount will, in this case, be

$$a=\left(1+\frac{r}{x}\right)^{nx}.$$

Expanding the right-hand member by the Binomial Theorem, we find

$$a=1+nx.\frac{r}{x}+\frac{nx(nx-1)}{1.2}.\frac{r^2}{x^2}+\frac{nx(nx-1)(nx-2)}{1.2.3}.\frac{r^3}{x^3}+\&c.$$

When $x=\infty$, this becomes

$$a=1+nr+\frac{n^2r^2}{1.2}+\frac{n^3r^3}{1.2.3}+\frac{n^4r^4}{1.2.3.4}+\&c.$$

Comparing this with formula (B), Art. 243, we have

$$a = e^{nr}.$$

Using the common logarithms, we have

$$\log. a = nr \times 0.4342944819.$$

This formula gives the logarithm of the amount of \$1 for n years at r per cent.

If we call $n = 1$, $r = 0.07$, we shall find 1.0725 nearly, for the amount. That is, the *instantaneous* compound interest of \$1 for 1 year, at 7 per cent. per *annum*, is $7\frac{1}{4}$ cents nearly. Hence, however often a person adds the interest to the principal, to form a new principal, he cannot make more than $7\frac{1}{4}$ per cent., when the rate is 7 per cent. per annum. If we take the rate per cent. per annum, at $6\frac{23}{30}$, and compute the instantaneous compound interest on \$1 for 1 year, it will be just the same as the simple interest of \$1 for the same time at 7 per cent.

(253.) Before closing this chapter, we will show how formulas 17, 18, 19, and 20 of Geometrical Progression were found.

By taking the logarithm of both members of No. 1, as given in the table under Art. 178, we have

$$\log. l = \log. a + (n-1)\log. r.$$

This gives

$$n - 1 = \frac{\log. l - \log. a}{\log. r},$$

or,

$$n = \frac{\log. l - \log. a}{\log. r} + 1,$$

which agrees with No. 17.

No. 5 is readily put in the following form :

$$a + (r-1)s = ar^n.$$

Taking the logarithms, we have

$$\log.\ [a+(r-1)s]=\log.\ a+n\log.\ r,$$

from which we readily get

$$n=\frac{\log.\ [a+(r-1)s]-\log.\ a}{\log.\ r},$$

which agrees with No. 18.

No. 12 may take the following form :

$$a(s-a)^{n-1}=l(s-l)^{n-1},$$

which, in logarithms, is

$$\log.\ a+(n-1)\log.\ (s-a)=\log.\ l+(n-1)\log.(s-l),$$

which gives

$$n=\frac{\log.\ l-\log.\ a}{\log.(s-a)-\log.\ (s-l)}+1,$$

which agrees with No. 19.

Again, No. 16 may be written as follows :

$$r(s-l)r^{n-1}-sr^{n-1}+l=0,$$

which is readily reduced to

$$[rl-(r-1)s]r^{n-1}=l.$$

Taking the logarithms, we have

$$\log.\ [rl-(r-1)s]+(n-1)\log.\ r=\log.\ l.$$

From this we find

$$n=\frac{\log.\ l-\log.\ [rl-(r-1)s]}{\log.\ r}+1,$$

which is the same as No. 20.

CHAPTER X.

GENERAL PROPERTIES OF EQUATIONS.

(254.) Any number or quantity which, when substituted for the unknown quantity in an equation, satisfies the equation, is called a root of that equation.

If the general algebraic equation,

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots+A_{n-1}x+A_n=0, \qquad (1)$$

is satisfied by making $x=a_1$, then a_1 is a root of equation (1).

Substituting a_1 for x in (1), we get

$$a_1^n+A_1a_1^{n-1}+A_2a_1^{n-2}\ldots\ldots+A_{n-1}a_1+A_n=0. \qquad (2)$$

Subtracting (2) from (1), we have

$$\left.\begin{array}{r} x^n-a_1^n+A_1(x^{n-1}-a_1^{n-1})+A_2(x^{n-2}-a_1^{n-2}) \\ \ldots\ldots A_{n-1}(x-a_1)=0. \end{array}\right\} \qquad (3)$$

We know that each of the expressions

$$\begin{array}{c} x^n-a_1^n, \\ x^{n-1}-a_1^{n-1}, \\ x^{n-2}-a_1^{n-2}, \\ \ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \\ x-a_1, \end{array}$$

is divisible by $x-a_1$; consequently, the left-hand member of (3) is divisible by $x-a_1$.

Equation (3) does not differ from (1), since (3) was derived from (1) by subtracting from it equation (2), which is equal to 0. Therefore, equation (1) is also divisible by $x-a_1$; hence the following property:

(255.) *If a_1 is a root of the general algebraic equation*

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots+A_{n-1}x+A_n=0,$$

then its left-hand member will be divisible by $x-a_1$.

As an example, suppose 3 is a root of the equation

$$x^3-7x^2+36=0.$$

Now, by the above property this equation must be divisible by $x-3$.

Actually performing the division, we have

$$\begin{array}{l|l}
x^3-7x^2+36 & x-3 \text{ divisor.} \\
x^3-3x^2 & \overline{} \\
\hline
\quad -4x^2 & x^2-4x-12 \text{ quotient.} \\
\quad -4x^2+12x & \\
\hline
\qquad -12x+36 & \\
\qquad -12x+36 & \\
\hline
\qquad\qquad 0. &
\end{array}$$

(256.) If we divide our general equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots+A_{n-1}x+A_n=0, \qquad (1)$$

by $x-a_1$, we shall obtain for a quotient, a new equation of one degree less than equation (1), which may be represented as follows:

$$x^{n-1}+B_1x^{n-2}+B_2x^{n-3}\ldots+B_{n-2}x+B_{n-1}=0. \qquad (2)$$

This equation must also have a root, which we will represent by a_2. Again, dividing (2) by $x-a_2$, we shall obtain

a new equation one degree less than (2), and consequently two degrees less than (1). Let this new equation be represented by

$$x^{n-2}+C_1x^{n-3}+C_2x^{n-4}\ldots\ldots+C_{n-3}x+C_{n-2}=0. \quad (3)$$

If a_3 is a root of equation (3), we can divide it by $x-a_3$, we shall thus find a new equation of three degrees less than equation (1). If we continue in this way, we shall, after n divisions, obtain an equation whose degree $=0$; therefore, equation (1), is composed of n factors.

$$x-a_1;\ x-a_2;\ x-a_3;\ \&c.$$

Hence, we have the following property:

(257.) *If $a_1, a_2, a_3, \ldots\ldots a_n$, denote the n roots of our general equation of the nth degree, then this equation will take the following form:*

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_{n-1})(x-a_n)=0.$$

This equation is verified by making either of the n factors $=0$; that is, by making $x=a_1$, or $x=a_2$, or $x=a_3$, &c., from which we infer, *that every equation of the nth degree, has n roots.*

(258.) It does not however follow, that all the roots a_1, a_2, a_3, a_4, &c., are different, since two or more of them may be equal, but still, their number must be n, since there are n factors.

(259.) If all the roots $a_1, a_2, a_3, \ldots\ldots a_n$ are negative, then each factor of the equation

$$(x+a_1)(x+a_2)(x+a_3)\ldots\ldots(x+a_{n-1})(x+a_n)=0,$$

will be positive, consequently each term of its equivalent value

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots\ldots\ldots+A_{n-1}x+A_n=0$$

will be positive.

If the roots are all positive, then will the terms of

$$(x-a_1)(x-a_2)(x-a_3)\ldots\ldots.(x-a_{n-1})(x-a_n)=0,$$

when expanded, be alternately positive and negative.

(260.) *Hence, if the terms of any equation are neither all positive, nor alternately positive and negative, that equation must contain both positive and negative roots.*

(261.) Reasoning after this manner, *Harriot* has shown

That every equation whose roots are possible, has as many changes of signs from + to —, or from — to +, as there are positive roots; and as many continuations of the same signs from + to +, or from — to —, as there are negative roots.

(262.) If, as we have already supposed, the n roots of an equation of the nth degree be denoted by $a_1, a_2, a_3, \ldots. a_n$, we can put the equation under the following form:

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)\ldots.(x-a_n)=0. \qquad (1)$$

Let us suppose $a_1 > a_2$; $a_2 > a_3$; $a_3 > a_4$; and so of the rest.

If a quantity b greater than a_1 be substituted for x in (1), the result will be positive, since all the factors will then be positive.

If a quantity c less than a_1, but greater than a_2, be substituted for x, the factors will be all positive except one, and consequently the result will be negative.

If a quantity d less than a_2, but greater than a_3, be substituted for x, all the factors except two will be positive; and since two negative factors produce a positive product, the result must be positive.

By following out this plan of reasoning, we deduce the following property:

(263.) *If two quantities be successively substituted for x in any equation, and give results affected with* DIFFERENT SIGNS, *there must be an odd number of roots between these quantities.*

But if the two quantities when substituted for x give results affected with the SAME SIGNS, *there must be either no root, or else an even number of roots between these quantities.*

EXAMPLES.

1. Find the first figure of one of the roots of the equation

$$x^3+1.5x^2+0.3x-46=0.$$

If we substitute 3 for x, the result will be -4.6, a negative quantity. If we substitute 4 for x, the result will be 43.2, a positive quantity. Therefore, the first figure of the root sought must be 3.

2. Find the first figure of one of the roots of the equation

$$x^4+3x^3+2x^2+6x-148=0.$$

Putting 2 for x, the result is -88, and putting 3 for x, we get 50, $\therefore$ the first figure of the root sought is 2.

3. Find the first figure of one of the roots of the equation

$$x^3-17x^2+54x-350=0.$$

In this example, the two consecutive numbers between which there is a root, are 10 and 20, therefore, the first figure of the root sought is 1 in the ten's place.

(264.) By actual multiplication, we find

$$(x-a_1)(x-a_2)=x^2\left.\begin{matrix}-a_1\\-a_2\end{matrix}\right\}x+a_1a_2,$$

$$(x-a\ )(x-a_2)(x-a_3)=x^3\left.\begin{matrix}-a_1\\-a_2\\-a_3\end{matrix}\right\}x^2\left.\begin{matrix}+a_1a_2\\+a_1a_3\\+a_2a_3\end{matrix}\right\}x-a_1a_2a_3,$$

$$(x-a_1)(x-a_2)(x-a_3)(x-a_4)=x^4 \left.\begin{array}{l}-a_1\\-a_2\\-a_3\\-a_4\end{array}\right\} x^3$$

$$\left.\begin{array}{l}+a_1a_2\\+a_1a_3\\+a_1a_4\\+a_2a_3\\+a_2a_4\\+a_3a_4\end{array}\right\} x \qquad \left.\begin{array}{l}-a_1a_2a_3\\-a_1a_2a_4\\-a_1a_3a_4\\-a_2a_3a_4\end{array}\right\} x+a_1a_2a_3a_4,$$

&c., &c., &c.

By carefully examining the above results, we discover the following properties :

(265.) *The coefficient of x in the first term is always* 1.

The coefficient of the second term, is the sum of all the roots with their signs changed.

The coefficient of the third term, is the sum of all the products of the roots taken two at a time.

The coefficient of the fourth term, is the sum of all the products of the roots with their signs changed, taken three at a time.

And so on for the succeeding terms, until we reach the last term, which is independant of x, and is equal to the continued product of all the roots, with their signs changed.

(266.) The general form of an imaginary or impossible root of an equation is $a+\sqrt{-b}$.

The only factor which will render $a+\sqrt{-b}$ rational, is $a-\sqrt{-b}$.

We have just seen, that the last term of our general equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots A_{n-1}x+A_n=0.$$

is composed of the continued product of all its roots.

Hence, if $a+\sqrt{-b}$ is a root of this equation, then also will $a-\sqrt{-b}$ be a root, unless A_n is imaginary.

In the same way, if $a'+\sqrt{-b'}$ is a root, then will $a_1-\sqrt{-b'}$ be a root, and so for other imaginary roots. From this we infer the following properties:

(267.) *Every equation has an even number of impossible roots, or else none at all.*

An equation of an even degree may have all its roots impossible; but if they are not all impossible, two of them at least are possible.

If all the roots of an equation are impossible, then whatever values are substituted for x in that equation, the results will always be affected with the same signs.

An equation of an odd degree has at least one real root.

(268.) If we divide both members of the identical equation

$$\left.\begin{matrix} x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots A_{n-1}x+A_n= \\ (x-a_1)(x-a_2)(x-a_3)\ldots\ldots(x-a_n) \end{matrix}\right\}$$

by x^n, we shall obtain

$$\left.\begin{matrix} 1+\frac{A_1}{x}+\frac{A_2}{x^2}\ldots\ldots\frac{A_{n-1}}{x^{n-1}}+\frac{A_n}{x^n}= \\ \left(1-\frac{a_1}{x}\right)\left(1-\frac{a_2}{x}\right)\left(1-\frac{a_3}{x}\right)\ldots\ldots\left(1-\frac{a^n}{x}\right). \end{matrix}\right\}$$

Taking the logarithms of both members, we find

$$\log.\left\{1+\frac{A_1}{x}+\frac{A_2}{x^2}\cdots\frac{A_{n-1}}{x^{n-1}}+\frac{A_n}{x^n}\right\}=\left.\begin{array}{l}+\log.\left(1-\frac{a_1}{x}\right)\\ +\log.\left(1-\frac{a_2}{x}\right)\\ +\log.\left(1-\frac{a_3}{x}\right)\\ \cdots\cdots\cdots\cdots\\ \cdots\cdots\cdots\cdots\\ +\log.\left(1-\frac{a_n}{x}\right).\end{array}\right\}\text{(A)}$$

If we actually take the logarithm of the left-hand member of (A), by formula (C), Art. 237, where

$$\frac{A_1}{x}+\frac{A_2}{x^2}\cdots\cdots+\frac{A_{n-1}}{x^{n-1}}+\frac{A_n}{x^n}$$

is put for n, we shall obtain

$$\frac{A_1}{x}\;\left.\begin{array}{l}+A_2\\ -\frac{1}{2}A_1^2\end{array}\right\}\frac{1}{x^2}\;\left.\begin{array}{l}+A_3\\ -A_1A_2\\ +\frac{1}{3}A_1^3\end{array}\right\}\frac{1}{x^3}\;\left.\begin{array}{l}+A_4\\ -A_1A_3\\ -\frac{1}{2}A_2^2\\ +A_1^2A_2\\ -\frac{1}{4}A_1^4\end{array}\right\}\frac{1}{x^4}+\text{\&c.}\qquad\text{(B)}$$

By taking the logarithms of the terms of the right-hand member of (A), we get

$$\left.\begin{array}{l}-(a_1+a_2+a_3\ldots\ldots a_n)\frac{1}{x}\\ -\frac{1}{2}(a_1^2+a_2^2+a_3^2\ \ldots\ldots a_n^2)\frac{1}{x^2}\\ -\frac{1}{3}(a_1^3+a_2^3+a_3^3\ldots\ldots a_n^3)\frac{1}{x^3}\\ -\frac{1}{4}(a_1^4+a_2^4+a_3^4\ldots\ldots a_n^4)\frac{1}{x^4}\\ \text{\&c.,}\qquad\qquad \text{\&c.}\end{array}\right\}\qquad\text{(C)}$$

By equating the coefficients of the like powers of x, in (B) and (C), we find the following interesting properties:

$$\left.\begin{aligned}
&a_1+a_2+a_3\ldots\ldots a_n=-A_1,\\
&a_1^2+a_2^2+a_3^2\ldots\ldots a_n^2=A_1^2-2A_2,\\
&a_1^3+a_2^3+a_3^3\ldots\ldots a_n^3=-A_1^3+3A_1A_2-3A_3,\\
&a_1^4+a_2^4+a_3^4\ldots\ldots a_n^4=\\
&\qquad\qquad A_1^4-4A_1^2A_2+4A_1A_3+2A_2^2-4A_4,\\
&\quad\&c.,\qquad\qquad\qquad\qquad \&c.
\end{aligned}\right\}\quad(D)$$

(269.) These relations make known the sum of the mth powers of all the roots of an equation in terms of its coefficients.

(270.) If we suppose the general equation is deprived of its second term, or which amounts to the same thing, if we suppose $A_1=0$. the above results of (D) will become

$$\left.\begin{aligned}
&a_1+a_2+a_3\ldots\ldots a_n=0,\\
&a_1^2+a_2^2+a_3^2\ldots\ldots a_n^2=-2A_2,\\
&a_1^3+a_2^3+a_3^3\ldots\ldots a_n^3=-3A_3,\\
&a_1^4+a_2^4+a_3^4\ldots\ldots a_n^4=2A_2^2-4A_4,\\
&\quad\&c.,\qquad\qquad\qquad \&c.
\end{aligned}\right\}\quad(E)$$

TRANSFORMATIONS OF EQUATIONS.

(271.) We will resume our general equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots+A_{n-1}x+A_n=0. \qquad (1)$$

If in this equation we suppose $x=u+x'$, u being a new unknown quantity, and x' an indeterminate quantity, we shall have

$$\left.\begin{aligned}
&(u+x')^n+A_1(u+x')^{n-1}+A_2(u+x')^{n-2}\\
&\qquad\ldots\ldots A_{n-1}(u+x')+A_n=0,
\end{aligned}\right\}\quad(2)$$

which, when expanded by the *Binomial Theorem*, becomes

$$u^n + \left.\begin{matrix} nx' \\ +A_1 \end{matrix}\right\} u^{n-1} \left.\begin{matrix} +n.\frac{n-1}{2}x'^2 \\ +(n-1)A_1x' \\ +A_2 \end{matrix}\right\} u^{n-2} \ldots \left.\begin{matrix} +x'^n \\ +A_1x'^{n-1} \\ +A_2x'^{n-2} \\ \cdots\cdots \\ \cdots\cdots \\ +A_{n-1}x' \\ +A_n \end{matrix}\right\} = 0. \quad (3)$$

Now, since x' is wholly arbitrary, we are able to give it such a value as to satisfy this condition $nx' + A_1 = 0$; which is done by making $x' = -\frac{A_1}{n}$.

This value of x' substituted in (3), will give an equation of the following form :

$$u^n + B_2u^{n-2} + B_3u^{n-3} \ldots\ldots B_{n-1}u + B_n = 0, \qquad (4)$$

which is deprived of its second term.

(272.) *Hence, to cause the second term of an equation to disappear, we must replace the unknown by a new unknown augmented by the coefficient of the second term with its sign changed, and divided by the number denoting the degree of the equation.*

EXAMPLES.

1. Transform the quadratic equation

$$x^2 + A_1x + A_2 = 0,$$

into a new equation wanting its second term.

Assume $x = u - \frac{A_1}{2}$, and it will become

$$\left(u - \frac{A_1}{2}\right)^2 + A_1\left(u - \frac{A_1}{2}\right) + A = 0,$$

this, when reduced, becomes

$$u^2 - \left(\frac{A_1^2}{4} - A^2\right) = 0,$$

or, by transposing,

$$u^2 = \frac{A_1^2}{4} - A_2 \quad \therefore u = \pm\sqrt{\frac{A_1^2}{4} - A_2},$$

and, $$x = u - \frac{A_1}{2} = -\frac{A_1}{2} \pm \sqrt{\frac{A_1^2}{4} - A_2}.$$

The same result as was obtained by the direct solution of the above equation under Art. 151, formula (D).

2. Transform the cubic equation

$$x^3 + A_1x^2 + A_2x + A_3 = 0,$$

into a new equation, wanting its second term.

Assuming $x = u - \frac{A_1}{3}$, we get

$$\left(u - \frac{A_1}{3}\right)^3 + A_1\left(u - \frac{A_1}{3}\right)^2 + A_2\left(u - \frac{A_1}{3}\right) + A_3 = 0,$$

which, when expanded and reduced, gives

$$u^3 + B_2u + B_3 = 0,$$

where $$B_2 = -\frac{A_1^2}{3} + A_2,$$

$$B_3 = \frac{2A_1^3}{27} - \frac{A_1A_2}{3} + A_3.$$

We might proceed in this way for the transformation of equations of higher degrees, but its easy to see that this method would be very lengthy and complicated for such equations, we shall therefore seek some law by which these transformations can be made with less labor.

(273.) If in the general equation

$$x^n + A_1x^{n-1} + A_2x^{n-2} \ldots\ldots + A_{n-1}x + A_n = 0 = X.$$

we substitute $x'+u$ for x, and imitate the operations of Art. 271, we shall have

$$\left.\begin{array}{l}
\begin{array}{ll}
\begin{array}{l} x'^n \\ +A_1x'^{n-1} \\ +A_2x'^{n-2} \\ \cdots\cdots \\ \cdots\cdots \\ +A_{n-1}x' \\ +A_n \end{array} &
\left.\begin{array}{l} +nx'^{n-1} \\ +(n-1)A_1x'^{n-2} \\ +(n-2)A_2x'^{n-3} \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ +A_{n-1} \\ \end{array}\right\} u
\end{array} \\
\\
\left.\begin{array}{l} +n\,.\,\dfrac{n-1}{2}\,x'^{n-2} \\ +(n-1).\dfrac{n-2}{2}\,A_1x'^{n-3} \\ +(n-2).\dfrac{n-3}{2}\,A_2x'^{n-4} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \end{array}\right\} u^2+\ldots+u^n
\end{array}\right\}=0.$$

If in the above transformation, we put X' for the coefficient of u^0, or which is the same thing, for the sum of the terms independent of u. Also, put X'' for the coefficient of u, and $\frac{X'''}{2}$ for the coefficient of u^2, $\frac{X''''}{2.3}$ for the coefficient of u^3, and so on, we shall have

$$\begin{array}{l}
X=x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots\ldots\ldots A_{n-1}x+A_n, \\
X'=x'^n+A_1x'^{n-1}+A_2x'^{n-2}\ldots\ldots\ldots\ldots A_{n-1}x'+A_n, \\
X''=nx'^{n-1}+(n-1)A_1x'^{n-2}+(n-2)A_2x'^{n-3}\ldots\ldots A_{n-1}, \\
X'''=n(n-1)x'^{n-2}+(n-1)(n-2)A_1x'^{n-3}+\ldots\ldots\ldots \\
X''''=n(n-1)(n-2)x'^{n-3}+(n-1)(n-2)(n-3)A_1x'^{n-4}\ldots \\
\qquad \&c., \qquad\qquad\qquad\qquad \&c.
\end{array}$$

If we examine the above expressions, we shall discover the following law:

X' is derived from the general equation X, by simply changing x into x'.

X'' is derived from X' by multiplying each of the terms of X' by the exponent of x' in that term, and diminishing this exponent by a unit.

X''' is derived from X'' in the same manner as X'' was derived from X'.

And, in general, a coefficient of any rank, in the above transformed equation, is formed by means of the preceding, by multiplying each term of the preceding by its exponent, and dividing the product by the number of coefficients which precedes the terms sought, and diminishing the exponent by a unit.

(274.) The polynomial X'' is called the *first derived polynomial* of X'.

The polynomial of X''' is called the *second derived polynomial* of X'; and so on for the succeeding polynomials.

(275.) We will add a few examples to illustrate the above law.

1. Transform the equation

$$x^4-12x^3+17x^2-9x+7=0,$$

into an equation wanting its second term.

By Art. 272, we must substitute $u+\frac{12}{4}=u+3$, or $3+u$ for x; this transformed by Art. 273, will be of this form:

$$X'+X''u+\frac{X'''}{2}u^2+\frac{X''''}{2.3}.u^3+u^4=0.$$

Now, by the above law, we find

$$X' = (3)^4 - 12(3)^3 + 17(3)^2 - 9(3)^1 + 7 = -110,$$
$$X'' = 4(3)^3 - 36(3)^2 + 34(3)^1 - 9 = -123,$$
$$\frac{X'''}{2} = 6(3)^2 - 36(3)^1 + 17 = -37,$$
$$\frac{X''''}{2.3} = 4(3)^1 - 12 = 0.$$

Hence, our transformed equation is

$$u^4 - 37u^2 - 123u - 110 = 0.$$

2. Transform

$$x^5 - 10x^4 + 7x^3 + 4x - 9 = 0,$$

into a new equation wanting its second term.

Proceeding as above, we find

$$X' = (2)^5 - 10(2)^4 + 7(2)^3 + 4(2)^1 - 9 = -73,$$
$$X'' = 5(2)^4 - 40(2)^3 + 21(2)^2 + 4 = -152,$$
$$\frac{X'''}{2} = 10(2)^3 - 60(2)^2 + 21(2)^1 = -118,$$
$$\frac{X''''}{2.3} = 10(2)^2 - 40(2)^1 + 7 = -33,$$
$$\frac{X'''''}{2.3.4} = 5(2)^1 - 10 = 0.$$

Hence, our transformed equation is

$$u^5 - 33u^3 - 118u^2 - 152u - 73 = 0.$$

3. Transform

$$3x^3 + 15x^2 + 25x - 3 = 0,$$

into an equation wanting its second term.

Dividing each term by 3, in order to make it agree with the general equation, we get

$$x^3 + 5x^2 + \frac{25}{3}x - 1 = 0.$$

Now, in order to make the second term disappear, we must, by Art. 272, substitute $-\frac{5}{3}+u$ for x.

Hence,

$$X' = \left(-\frac{5}{3}\right)^3 + 5\left(-\frac{5}{3}\right)^2 + \frac{25}{3}\left(-\frac{5}{3}\right) - 1 = -\frac{152}{27},$$

$$X'' = 3\left(-\frac{5}{3}\right)^2 + 10\left(-\frac{5}{3}\right) + \frac{25}{3} \quad = \quad 0,$$

$$\frac{X'''}{2} = 3\left(-\frac{5}{3}\right) + 5 \quad = \quad 0.$$

Hence, the transformed sought, is

$$u^3 - \frac{152}{27} = 0.$$

In this example, the third term vanished at the same time as the second.

4. Transform

$$4x^3 - 5x^2 + 7x - 9 = 0,$$

into a new equation, of which the roots shall exceed by a unit, each of the corresponding roots of the given equation.

We must assume $u = x + 1$ or $x = u - 1$, which gives

$$X' = 4\,(-1)^3 - 5\,(-1)^2 + 7(-1)^1 - 9 = -25,$$

$$X'' = 12(-1)^2 - 10(-1)^1 + 7 \quad = \quad 29,$$

$$\frac{X'''}{2} = 12(-1)^1 - 5 \quad = -17,$$

$$\frac{X''''}{2.3} = 4.$$

Hence, the transformed equation is

$$4u^3 - 17u^2 + 29u - 25 = 0.$$

(276.) The derived polynomials possess some remarkable properties, which we will develop.

Let X, or

$$x^n + A_1 x^{n-1} + A_2 x^{n-2} + \ldots\ldots + A_{n-1} x + A_n = 0, \quad (1)$$

have $a_1, a_2, a_3, \ldots\ldots\ldots\ldots a_{n-1}, a_n$, for its n roots, we shall then have by Art. 257, the identical equation

$$\left.\begin{aligned} x^n + A_1 x^{n-1} + \ldots\ldots\ldots\ldots + A_{n-1} x + A_n = \\ (x - a_1)(x - a_2) \ldots\ldots\ldots\ldots (x - a_{n-1})(x - a_n). \end{aligned}\right\} \quad (2)$$

In (2) change x into $x + u$, and it will become

$$\left.\begin{aligned} (x+u)^n + A_1 (x+u)^{n-1} \ldots\ldots A_{n-1}(x+u) + A_n = \\ [u + (x - a_1)][u + (x - a_2)] \ldots\ldots [u + (x - a_n)]. \end{aligned}\right\} \quad (3)$$

The left-hand member of (3), by Art. 273, is

$$X + X'u + \frac{X''}{2} u^2 + \ldots\ldots u^n. \quad (4)$$

If we should actually perform the multiplication of the factors of the right-hand member, we should find, by paying attention to the properties under Art. 265, that the part independent of u is equal to

$$(x - a_1)(x - a_2)(x - a_3) \ldots\ldots (x - a_{n-1})(x - a_n). \quad (5)$$

The coefficient of u will equal the sum of the products of all the terms $x - a_1$, $x - a_2$, $x - a_3, \ldots\ldots$ taken $n - 1$ at a time.

The coefficient of u^2 will equal the sum of the products of the same terms taken $n - 2$ at a time.

Hence, by equating the coefficients of the like powers of u, in the two numbers of (3), we have

$$\left.\begin{array}{l}
X=(x-a_1)(x-a_2)(x-a_3)\ldots(x-a_{n-1})(x-a_n) \\
X'=\dfrac{X}{x-a_1}+\dfrac{X}{x-a_2}+\dfrac{X}{x-a_3}+\ldots\dfrac{X}{x-a_n} \\
X''=\left.\begin{array}{l}\dfrac{X}{(x-a_1)(x-a_2)}+\dfrac{X}{(x-a_1)(x-a_3)}+\ldots\ldots \\ \qquad\qquad +\dfrac{X}{(x-a_{n-1})(x-a_n)}.\end{array}\right\} \\
\qquad \&c. \qquad\qquad\qquad \&c.
\end{array}\right\} \quad (A)$$

EQUATIONS HAVING EQUAL ROOTS.

(277.) Let X denote the first member of the equation

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots+A_{n-1}x+A_n=0, \qquad (1)$$

and suppose m factors equal to $x-a$, m' factors equal to $x-b$, m'' factors equal to $x-c$, &c.; also, that it contains the simple factors $x-p, x-q, x-r$, &c., then we shall have

$$\left.\begin{array}{l} X=(x-a)^m(x-b)^{m'}(x-c)^{m''}\ldots \\ (x-p)(x-q)(x-r)\ldots=0. \end{array}\right\} \qquad (2)$$

Calling X' the first derived of X, we shall, by (A), Art. 260, have

$$X'=\frac{mX}{x-a}+\frac{m'X}{x-b}+\frac{m''X}{x-c}+\ldots\frac{X}{x-p}+\frac{X}{x-q}+\frac{X}{x-r}+\ldots(3)$$

Hence, the greatest common divisor of X and X' is

$$D=(x-a)^{m-1}(x-b)^{m'-1}(x-c)^{m''-1}. \qquad (4)$$

(278.) From this we conclude, that when the equation $X=0$ has no equal roots, then the polynomials have no common measure.

(279.) If the greatest common divisor D, equation (4), is of the first degree, and equal to $x-h=0$, we conclude that equation $X=0$ has two roots equal to h. And in general, if

it is of the form $(x-h)^n=0$, then the equation has $n+1$ roots equal to h.

When it is of the form $x^2+A_1x+A_2=0$, we must find the two values of x by quadratics, which we will suppose to be k and k', so that the equation will have two roots $=k$, and two more $=k'$.

EXAMPLES.

1. Has the equation

$$2x^4-12x^3+19x^2-6x+9=0$$

any equal roots, and if so, what are they?

$$X=2x^4-12x^3+19x^2-6x+9,$$
$$X'=8x^3-36x^2+38x-6.$$

Now, by the method of Art. 50, we find the greatest common measure of X and X' to be

$$D=x-3.$$

Therefore, the above equation has two roots equal to 3.

Dividing its first member by

$$(x-3)^2=x^2-6x+9,$$

we find

$$\begin{array}{r|l} 2x^4-12x^3+19x^2-6x+9 & x^2-6x+9. \\ \underline{2x^4-12x^3+18x^2} & \overline{2x^2+1.} \\ x^2-6x+9 & \\ \underline{x^3-6x+9} & \\ 0. & \end{array}$$

The two roots of $2x^2+1=0$ are $x=\pm\sqrt{-\frac{1}{2}}$.

Hence, the four roots of the above equation are

$$3,\ 3,\ +\sqrt{-\tfrac{1}{2}},\ -\sqrt{-\tfrac{1}{2}}.$$

2. Find the equal roots of

$$x^5-2x^4+3x^3-7x^2+8x-3=0,$$

if it has any.

$$X = x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3,$$
$$X' = 5x^4 - 8x^3 + 9x^2 - 14x + 8.$$

Seeking the greatest common divisor of X and X', we find

$$D = x^2 - 2x + 1 = (x-1)^2,$$

hence, there are 3 roots equal to 1.

If we divide the value of X by

$$(x-1)^3 = x^3 - 3x^2 + 3x - 1$$

we shall obtain the quotient $x^2 + x + 3$.

The two roots of $x^2 + x + 3 = 0$, are $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-11}$, hence the five roots of

$$x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3 = 0,$$
$$1,\ 1,\ 1, -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-11}, -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-11}.$$

3. Find the roots of

$$x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4 = 0.$$

Proceeding as in the last example, we find

$$X = x^7 + 5x^6 + 6x^5 - 6x^4 - 15x^3 - 3x^2 + 8x + 4,$$
$$X' = 7x^6 + 30x^5 + 30x^4 - 24x^3 - 45x^2 - 6x + 8,$$
$$D = x^4 + 3x^3 + x^2 - 3x - 2.$$

Now, since the greatest common divisor D, surpasses the second degree, we cannot immediately resolve it.

If we apply the same process to D, as we have done to X, we shall find

$$D = x^4 + 3x^3 + x^2 - 3x - 2,$$
$$D' = 4x^3 + 9x^2 + 2x - 3 = \text{first derived of } D,$$
$$D'' = x + 1 = \text{greatest common divisor of } D \text{ and } D'.$$

Hence, D has two roots equal to -1. Dividing it by

$$(x+1)^2 = x^2 + 2x + 1,$$

we obtain the quotient $x^2 + x - 2$,
which equated to zero, gives $x = 1$, or $x = -2$.

Therefore, $D=(x+1)^2(x-1)(x+2)$,
and consequently, $X=(x+1)^3(x-1)^2(x+2)^2$
and the equation has three roots, $=-1$; two roots, $=1$; and two roots, $=-2$.

RECURRING EQUATIONS.

(280.) *A recurring equation* is one which remains the same when $\frac{1}{x}$ is substituted for x.

All recurring equations are of this form:

$$x^n+A_1x^{n-1}+A_2x^{n-2}\ldots\ldots+A_2x^2+A_1x+1=0, \quad (1)$$

where the coefficients of the terms equi-distant from the extremes are equal, because, if for x we substitute $\frac{1}{x}$, the above equation will become

$$\frac{1}{x^n}+\frac{A_1}{x^{n-1}}+\frac{A_2}{x^{n-2}}\ldots\ldots\ldots\frac{A_2}{x_2}+\frac{A_1}{x}+1=0, \quad (2)$$

which, when cleared of fractions by multiplying by x^n, becomes

$$1+A_1x+A_2x^2+\ldots\ldots\ldots+A_2x^{n-2}+A_1x^{n-1}+x^n=0, \quad (3)$$

which is just the same as equation (1), only the terms are taken in a reverse order.

From the above definition of a recurring equation, we know, that if a_1 is one of the roots, then will $\frac{1}{a_1}$ also be a root of this equation.

Hence, recurring equations are sometimes called *reciprocal equations*.

(281.) A recurring equation of an odd degree can in general be represented by

$$x^{2n+1} \pm A_1 x^{2n} \pm A_2 x^{2n-1} \pm \ldots \pm A_2 x^2 \pm A_1 x \pm 1 = 0. \quad (4)$$

Now, if the corresponding coefficients have the same sign, $x = -1$ will satisfy (4), but if the corresponding coefficients have contrary signs, then $x = 1$ will satisfy (4).

(282.) *Hence, -1 or $+1$ is always one root of a recurring equation of an odd degree; consequently, by Art. 255 we know that a recurring equation of an odd degree is divisible by $x+1$, or by $x-1$; and the quotient will be a recurring equation of one degree lower, and consequently of an even degree.*

(283.) The general form of a recurring equation of an even degree is

$$x^{2n} + A_1 x^{2n-1} + A_2 x^{n-2} + \ldots + A_2 x^2 + A_1 x + 1 = 0. \quad (5)$$

This divided by x^n, becomes

$$x^n + A_1 x^{n-1} + A_2 x^{n-2} + \ldots + \frac{A_2}{x^{n-2}} + \frac{A_1}{x^{n-1}} + \frac{1}{x_n} = 0, \quad (6)$$

which becomes, by bringing the terms of equal coefficients together,

$$x^n + \frac{1}{x^n} + A_1\left(x^{n-1} + \frac{1}{x^{n-1}}\right) + A_2\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + \&c. = 0. \quad (7)$$

If we expand $\left(x^n + \frac{1}{x^n}\right) \times \left(x + \frac{1}{x}\right)$, we shall obtain this identical equation,

$$\left(x^n + \frac{1}{x^n}\right) \times \left(x + \frac{1}{x}\right) = x^{n+1} + \frac{1}{x^{n+1}} + x^{n-1} + \frac{1}{x^{n-1}}. \quad (8)$$

By transposing, we have

$$x^{n+1} + \frac{1}{x^{n+1}} = \left(x^n + \frac{1}{x^n}\right)z - \left(x^{n-1} + \frac{1}{x^{n-1}}\right), \quad (9)$$

where $z = x + \frac{1}{x}$.

If in formula (9) we suppose successively

$$n = 1, 2, 3, 4, 5, \ldots\ldots,$$

we shall find

$$\left.\begin{aligned}
x^2+\frac{1}{x^2} &= \left(x+\frac{1}{x}\right)z - \left(x^0+\frac{1}{x^0}\right) = z^2-2,\\
x^3+\frac{1}{x^3} &= \left(x^2+\frac{1}{x^2}\right)z - \left(x+\frac{1}{x}\right) = z^3-3z,\\
x^4+\frac{1}{x^4} &= \left(x^3+\frac{1}{x^3}\right)z - \left(x^2+\frac{1}{x^2}\right) = z^4-4z^2+2,\\
x^5+\frac{1}{x^5} &= \left(x^4+\frac{1}{x^4}\right)z - \left(x^3+\frac{1}{x^3}\right) = z^5-5z^3+5z,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \quad (A)$$

These values of $x+\frac{1}{x}$; $x^2+\frac{1}{x^2}$; $x^3+\frac{1}{x^3}$; &c., in terms of z, being substituted in the general recurring equation of an even degree, will give an equation in terms of z of but half that degree.

(284.) From Art. 281, we know that a recurring equation of the degree $2n+1$, can be immediately reduced to a recurring equation of the degree $2n$, by dividing by $x+1$, or $x-1$. Consequently a recurring equation of the degree $2n+1$ can be reduced to an equation of the nth degree.

Suppose, for example, we wish to find the five roots of the recurring equation

$$x^5-11x^4+17x^3+17x^2-11x+1=0. \qquad (1)$$

Since this is a recurring equation of an odd degree, and the corresponding coefficients have the same sign, it follows, by Art. 281, that one of its roots is -1. Dividing this equation by $x+1$, we obtain for a quotient this new recurring equation of the fourth degree.

$$x^4-12x^3+29x^2-12x+1=0. \qquad (2)$$

Dividing this by x^2, and reducing the result to the form of (9), Art. 283, we have

$$x^2+\frac{1}{x^2}-12\left(x+\frac{1}{x}\right)+29=0. \qquad (3)$$

Substituting, in (3), for $x^2+\frac{1}{x^2}$, $x+\frac{1}{x}$, their values in terms of z, as given by group (A), Art. 283, we obtain

$$z^2-2-12z+29=0, \qquad (4)$$

or

$$z^2-12z+27=0. \qquad (5)$$

Equation (5), solved by the usual rule for quadratics, gives

$$z=9, \text{ or } z=3. \qquad (6)$$

Taking the first value of z, we have

$$z=x+\frac{1}{x}=9, \text{ or } x^2-9x=-1. \qquad (7)$$

Solving (7) by quadratics, we find

$$x=\frac{9}{2}\pm\frac{1}{2}\sqrt{77} \qquad (8)$$

Taking the second value of z, we have

$$z=x+\frac{1}{x}=3, \text{ or } x^2-3x=-1. \qquad (9)$$

Equation (9) gives

$$x=\frac{3}{2}\pm\frac{1}{2}\sqrt{5}. \qquad (10)$$

Therefore, the five roots of the proposed equation are

$$-1, \frac{9+\sqrt{77}}{2}, \frac{9-\sqrt{77}}{2}, \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}.$$

If the numerator and denominator of the third root be each multiplied by $9+\sqrt{77}$, and the numerator and denominator of the fifth root, be each multiplied by $3+\sqrt{5}$, the roots will assume the following form:

$$-1, \frac{9+\sqrt{77}}{2}, \frac{2}{9+\sqrt{77}}, \frac{3+\sqrt{5}}{2}, \frac{2}{3+\sqrt{5}},$$

which shows that the third root is the reciprocal of the second, and the fifth is the reciprocal of the fourth.

BINOMIAL EQUATIONS.

(285.) *Binomial equations* are of this form:

$$y^n \pm a^n = 0;$$

in which, if we substitute ax for y, and divide the result by a^n, we shall obtain

$$x^n \pm 1 = 0,$$

for the general form of binomial equations.

(286.) If n is even, the equation $x^n + 1 = 0$; or, $x^n = -1$, gives for x the impossible expression $\sqrt[n]{-1}$; hence all the roots are imaginary. But the equation $x^n - 1 = 0$; or $x^n = 1$, gives $x = \sqrt[n]{1} = +1$, or -1; so that the equation has two real roots, and $n-2$ imaginary roots.

(287.) If n is odd, the equation $x^n + 1 = 0$; or $x^n = -1$; gives $x = \sqrt[n]{-1} = -1$; so that there is one real root and $n-1$ imaginary roots. But the equation $x^n - 1 = 0$; or $x^n = 1$, gives $x = \sqrt[n]{1} = +1$, so that, as before, we have one real root and $n-1$ imaginary roots.

(288.) If a is one of the imaginary roots of the binomial equation,

then will

$$\left.\begin{array}{rll} x^n = 1, & & (1) \\ (a^1)^n = a^n = 1, & & (2) \\ (a^2)^n = a^{2n} = 1^2 = 1, & & (3) \\ (a^3)^n = a^{3n} = 1^3 = 1, & & (4) \\ (a^4)^n = a^{4n} = 1^4 = 1, & & (5) \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots\ldots & & \end{array}\right\}$$

So that a^1, a^2, a^3, a^4, &c., satisfy the equation $x^n = 1$, when substituted for x. These quantities are therefore roots of the above equations.

Hence, if a is one of the imaginary roots of the equation $x^n = 1$, then any power of a, will also be an imaginary root.

From this it follows, that the roots $x^n = 1$, may be represented under an infinite variety of forms, each term in the following series being a root.

$$\left.\begin{array}{r} 1,\ a,\ a^2,\ a^3, \ldots\ldots a^{n-1}, \\ a^n,\ a^{n+1},\ a^{n+2}, \ldots\ldots a^{2n-1}, \\ a^{2n},\ a^{2n+1},\ a^{2n+2}, \ldots\ldots a^{3n-1}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (A)$$

(289.) When n is a prime number, the roots of the equation $x^n = 1$, are all contained in either of the expressions (A), for in each of these series of roots all the n terms will be different. But when n is a composite number, the roots of the equation are not all contained in either of the series (A), for some of them will be the same root under different forms, for suppose $n = p \times q$, and let $q > p$, then the first series of (A) is the same as

$$1, a, a^2, a^3, \ldots\ldots a^p, a^{p+1}, a^{p+2}, \ldots\ldots a^q, a^{q+1}, \ldots\ldots a^{pq-1}.$$

Now, since

$$a^{pq} = 1,\ a^p = \sqrt[q]{1} = 1\ ;\ \text{also}\ a^q = \sqrt[p]{1} = 1\ ;$$

therefore the terms 1, a^p, and a^q, are each equal to 1, and consequently, each must be the same root under different forms.

(290.) Suppose we have $x^{pp} = 1$, where $p =$ a prime. If we put $x^p = y$; then $y^p = 1$.

Now, suppose b is a root of $y^p = 1$, it will follow from Art. 288, that the p roots of $y^p = 1$ will be denoted by

$$1,\ b,\ b^2,\ b^3, \ldots\ldots\ldots\ldots b^{p-1}.$$

Hence, by substitution, we have

$$x^p - y = \left\{\begin{array}{ll} x^p - 1 = 0, & (1) \\ x^p - b = 0, & (2) \\ x^p - b^2 = 0, & (3) \\ \ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots & \\ x^p - b^{p-1} = 0. & (p) \end{array}\right\} \qquad \text{(B)}$$

The p roots of the first equation $x^p - 1 = 0$, have already been found to be

$$1,\ b,\ b^2,\ b^3, \ldots\ldots\ldots b^{p-1}.$$

If we make $x = z\sqrt[p]{b}$, the second equation of (B) will become

$$x^p - b = (z^p - 1) \times b = 0;$$

therefore the roots of $x^p - b = 0$, are equal to the roots of $z^p - 1 = 0$ multiplied by $\sqrt[p]{b}$.

Hence, the p roots of (2) are

$$\sqrt[p]{b},\ b\sqrt[p]{b},\ b^2\sqrt[p]{b}, \ldots\ldots\ldots\ldots b^{p-1}\sqrt[p]{b}.$$

Again, if we make $x = z\sqrt[p]{b^2}$, the third equation will become

$$x^p - b^2 = (z^p - 1) \times b^2 = 0;$$

therefore, the roots of $x^p - b^2 = 0$, are equal to the roots of $z^p - 1 = 0$, multiplied by $\sqrt[p]{b^2}$.

Hence, the p roots of (3) are

$$\sqrt[p]{b^2},\ b\sqrt[p]{b^2},\ b^2\sqrt[p]{b^2}, \ldots\ldots\ldots b^{p-1}\sqrt[p]{b^2}.$$

Proceeding in this way may find the following for the pp roots of $x^{pp} = 1$.

$$\left.\begin{array}{l}
1,\ b,\ b^2, \ldots\ldots b^{p-1}, \\
\sqrt[p]{b},\ b\sqrt[p]{b},\ b^2\sqrt[p]{b}, \ldots\ldots b^{p-1}\sqrt[p]{b}, \\
\sqrt[p]{b^2},\ b\sqrt[p]{b^2},\ b^2\sqrt[p]{b^2}, \ldots\ldots b^{p-1}\sqrt[p]{b^2}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\sqrt[p]{b^{p-1}},\ b\sqrt[p]{b^{p-1}},\ b^2\sqrt[p]{b^{p-1}}, \ldots\ldots b^{p-1}\sqrt[p]{b^{p-1}}.
\end{array}\right\} \quad (C)$$

(291.) Again, suppose we have $x^{pq} - 1 = 0$, where p and q are both primes.

If we put $x^p = y$, we shall have $y^q - 1 = 0$.

Let the q roots of this equation be

$$1,\ a,\ a^2,\ a^3, \ldots\ldots a^{q-1},$$

or which, by Art 288, is the same as

$$1,\ a^p,\ a^{2p},\ a^{3p} \ldots\ldots a^{(q-1)p},$$

then by substitution, we find

$$x^p - y = \left\{\begin{array}{ll}
x^p - 1 = 0, & (1) \\
x^p - a^p = 0, & (2) \\
x^p - a^{2p} = 0, & (3) \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\
\ldots\ldots\ldots\ldots\ldots\ldots & \ldots \\
x^p - a^{(q-1)p} = 0. & (q)
\end{array}\right\} \quad (D)$$

We will denote the values of x in $x^p - 1 = 0$, by

$$1,\ b,\ b^2,\ b^3, \ldots\ldots b^{p-1}.$$

If we make $x = az_{,}$, equation (2), of (D), will become

$$x^p - a^p = (z^p - 1)b^p = 0\ ;$$

therefore the roots of $x^p - a^p = 0$, are equal to the roots of $z^p - 1 = 0$ multiplied by a. And in a similar way we discover that the roots of $x^p - a^{2p} = 0$, are equal to the roots of $x^p - 1 = 0$ multiplied by a^2, and so for the other equations of (D).

Hence, the pq roots of $x^{pq} - 1 = 0$, are

$$\left.\begin{array}{l}1,\ b,\ b^2,\ b^3,\ \ldots\ldots b^{p-1},\\ a,\ ab,\ ab^2,\ ab^3,\ \ldots\ldots ab^{p-1},\\ a^2,\ a^2b,\ a^2b^2,\ a^2b^3,\ \ldots\ldots a^2b^{p-1},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ a^{q-1},\ a^{q-1}b,\ a^{q-1}b^2,\ a^{q-1}b^3,\ \ldots\ldots a^{q-1}b^{p-1}.\end{array}\right\}\quad (E)$$

As a particular case, suppose we wish the 15 roots of the equation $x^{15}-1=0$, or $x^{3.5}-1=0$.

In this case, $p=3$, and $q=5$; we must therefore seek the roots of $x^3-1=0$, and $x^5-1=0$.

We know, by Art. 287, that $x=1$ will satisfy each of the above equations; hence they are, by Art. 255, both divisible by $x-1$. If we effect the division, we shall have

$$x^2+x+1=0, \text{ and } x^4+x^3+x^2+x+1=0,$$

for the results; the first of these, $x^2+x+1=0$, being solved by quadratics, gives

$$x=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}, \text{ or } x=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}.$$

The other equation, $x^4+x^3+x^2+x+1=0$, is a recurring equation. Dividing it by x^2, we have

$$x^2+x+1+\frac{1}{x}+\frac{1}{x^2}=0,$$

or, which is the same thing,

$$\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0.$$

Substituting for $x^2+\frac{1}{x^2}$, and $x+\frac{1}{x}$, their values in terms of z, Art. 28?, we find

$$z^2+z-1=0.$$

This, solved by quadratics, gives

$$z=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{5}, \text{ or } z=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{5}.$$

Taking the first value of z, we have

$$z = x + \frac{1}{x} = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5}, \text{ or } x^2 - (\tfrac{1}{2}\sqrt{5} - \tfrac{1}{2})x = -1,$$

which, solved by quadratics, gives

$$x = \tfrac{1}{4}[\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}],$$
$$x = \tfrac{1}{4}[\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}].$$

Taking the second value of z, we have

$$z = x + \frac{1}{x} = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5}, \text{ or } x^2 + (\tfrac{1}{2}\sqrt{5} + \tfrac{1}{2})x = -1,$$

which, solved by quadratics, gives

$$x = -\tfrac{1}{4}[\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}],$$
or
$$x = -\tfrac{1}{4}[\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}].$$

In this case we have for the three roots of $x^3 - 1 = 0$, the following:

$$1 = 1,$$
$$b = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},$$
$$b^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

We have for the five roots of $x^5 - 1 = 0$, the following:

$$1 = 1,$$
$$a = \tfrac{1}{4}[\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}],$$
$$a^2 = -\tfrac{1}{4}[\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}],$$
$$a^3 = -\tfrac{1}{4}[\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}],$$
$$a^4 = \tfrac{1}{4}[\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}],$$

Consequently, the fifteen roots of $x^{15} - 1 = 0$, are

$$1 = 1,$$
$$a = \tfrac{1}{4}[\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}],$$
$$a^2 = -\tfrac{1}{4}[\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}],$$
$$a^3 = -\tfrac{1}{4}[\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}],$$
$$a^4 = \tfrac{1}{4}[\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}],$$

$$b = -\tfrac{1}{2}[1 - \sqrt{-3}],$$
$$ba = -\tfrac{1}{8}[1 - \sqrt{-3}] \times [\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}],$$
$$ba^2 = \tfrac{1}{8}[1 - \sqrt{-3}] \times [\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}],$$
$$ba^3 = \tfrac{1}{8}[1 - \sqrt{-3}] \times [\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}],$$
$$ba^4 = -\tfrac{1}{8}[1 - \sqrt{-3}] \times [\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}],$$
$$b^2 = -\tfrac{1}{2}[1 + \sqrt{-3}],$$
$$b^2a = -\tfrac{1}{8}[1 + \sqrt{-3}] \times [\sqrt{5} - 1 + \sqrt{-10 - 2\sqrt{5}}],$$
$$b^2a^2 = \tfrac{1}{8}[1 + \sqrt{-3}] \times [\sqrt{5} + 1 - \sqrt{-10 + 2\sqrt{5}}],$$
$$b^2a^3 = \tfrac{1}{8}[1 + \sqrt{-3}] \times [\sqrt{5} + 1 + \sqrt{-10 + 2\sqrt{5}}],$$
$$b^2a^4 = -\tfrac{1}{8}[1 + \sqrt{-3}] \times [\sqrt{5} - 1 - \sqrt{-10 - 2\sqrt{5}}].$$

If we extract the roots indicated, to 7 places of decimals, and reduce the results to their simplest forms, we shall have

$$1 = 1,$$
$$a = 0.3090170 + 0.9510565\sqrt{-1}, \qquad (3)$$
$$a^2 = -0.8090170 + 0.5877853\sqrt{-1}, \qquad (6)$$
$$a^3 = -0.8090170 - 0.5877853\sqrt{-1}, \qquad (6')$$
$$a^4 = 0.3090170 - 0.9510565\sqrt{-1}, \qquad (3')$$
$$b = -0.5000000 + 0.8660254\sqrt{-1}, \qquad (5)$$
$$ab = -0.9781476 - 0.2079117\sqrt{-1}, \qquad (7')$$
$$a^2b = -0.1045285 - 0.9945219\sqrt{-1}, \qquad (4')$$
$$a^3b = 0.9135454 - 0.4067366\sqrt{-1}, \qquad (1')$$
$$a^4b = 0.6691306 - 0.7431448\sqrt{-1}, \qquad (2')$$
$$b^2 = -0.5000000 - 0.8660254\sqrt{-1}, \qquad (5')$$
$$ab^2 = 0.6691306 + 0.7431448\sqrt{-1}, \qquad (2)$$
$$a^2b^2 = 0.9135454 + 0.4067366\sqrt{-1}, \qquad (1)$$
$$a^3b^2 = -0.1045285 + 0.9945219\sqrt{-1}, \qquad (4)$$
$$a^4b^2 = -0.9781476 + 0.2079117\sqrt{-1}, \qquad (7)$$

These imaginary roots are each of this form,

$$A \pm B\sqrt{-1}.$$

And in all cases,

$$A^2 + B^2 = 1.$$

For a complete and full discussion of the *Binomial Equation*, $x^n-1=0$, when n is a prime, the reader is referred to the 5th part, Vol. II, of *Legendre's Theorie des Nombres, 3d edition*, where he will find collected and demonstrated the many beautiful theorems on this subject, which were first published by *M. Gauss*, in his *Disquisitiones Arithmeticæ*.

(292.) Before closing this subject, it may not be amiss to apprise the student, that the solution of binomial equations are most readily found by the aid of Trigonometrical formula.

GENERAL SOLUTION OF AN EQUATION OF THE THIRD DEGREE

(293.) We have seen, Art. 272, that an equation of the third degree may be put under this form :

$$x^3+A_1x+A_2=0. \qquad (1)$$

If we assume $$x=y+z, \qquad (2)$$

we shall find $x^3=(y+z)^3=y^3+z^3+3yz(y+z)$
$$=y^3+z^3+3yz.x,$$
$$\therefore x^3-3yz.x-y^3-z^3=0. \qquad (3)$$

If we equate the coefficients of (3) with those of (1), we shall find

$$A_1=-3yz, \qquad (4)$$
$$A_2=-y^3-z^3. \qquad (5)$$

Which give $$yz=-\frac{A_1}{3}, \qquad (6)$$

$$y^3+z^3=-A_2. \qquad (7)$$

Cubing (6), we obtain

$$y^3z^3=-\frac{A_1^3}{27}. \qquad (8)$$

Squaring (7), we get $y^6+2y^3z^3+z^6=A_2^2. \qquad (9)$

Subtracting four times (8) from (9), and we have

$$y^6 - 2y^3z^3 + z^6 = A_2^2 + \frac{4A_1^3}{27}. \qquad (10)$$

Extracting the square root of (10), we find

$$y^3 - z^3 = \sqrt{A_2^2 + \frac{4A_1^3}{27}}. \qquad (11)$$

By adding and subtracting half of (11) to and from the half of (7), we find

$$y^3 = -\frac{A_2}{2} + \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}, \qquad (12)$$

$$z^3 = -\frac{A_2}{2} - \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}. \qquad (13)$$

Hence,

$$x = y + z =$$

$$\left\{-\frac{A_2}{2} + \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{A_2}{2} - \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}\right\}^{\frac{1}{3}}. \quad (A)$$

If we assume

$$\left.\begin{aligned} m &= \left\{-\frac{A_2}{2} + \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}\right\}^{\frac{1}{3}}, \\ n &= \left\{-\frac{A_2}{2} - \sqrt{\frac{A_2^2}{4} + \frac{A_1^3}{27}}\right\}^{\frac{1}{3}}, \end{aligned}\right\} \qquad (14)$$

the above value of x will become

$$x = m + n. \qquad (15)$$

Now, to obtain the other two roots, we will depress the equation

$$x^3 + A_1x + A_2 = 0,$$

by dividing it by $x - (m + n)$. See Art. 255.

OPERATION.

$$
\begin{array}{l|l}
x^3+A_1x+A_2 & x-(m+n) \\
x^3-(m+n)x^2 & \overline{x^2+(m+n)x+(m+n)^2+A_1.} \\
\hline
\quad (m+n)x^2+A_1x & \\
\quad (m+n)x^2-(m+n)^2x & \\
\hline
\qquad [(m+n)^2+A_1]x+A_2 & \\
\qquad [(m+n)^2+A_1]x-(m+n)^3-(m+n)A_1 & \\
\hline
\qquad\qquad (m+n)^3+(m+n)A_1+A_2. &
\end{array}
$$

As it regards this remainder, we see that since $m+n$ is a root of equation (1), it will be satisfied by substituting $m+n$ for x; making this substitution in (1), we find

$$(m+n)^3+(m+n)A_1+A_2=0,$$

which proves our remainder to vanish.

Hence, the true value of the depressed equation is

$$x^2+(m+n)x+(m+n)^2+A_1=0. \qquad (16)$$

This, solved by quadratics, gives

$$x=\frac{-(m+n)\pm\sqrt{-3(m+n)^2-4A_1}}{2}. \qquad (17)$$

So that equations (15) and (17) give the three roots of equation (1).

The two roots contained in (17) may be found from (15), as follows: Comparing equation (14) with (12) and (13), we find $y^3=m^3$, $z^3=n^3$; therefore, by Art. 288, we have

$$\left.\begin{array}{l} y=m, \\ y=am, \\ y=a^2m, \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} z=n, \\ z=an, \\ z=a^2n, \end{array}\right\}$$

where 1, a, a^2, are the three cube roots of 1; that is,

$$a = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}\,;\ a^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3}.$$

See example under Art. 291.

The only way in which we can combine the above six values of y and z, so that at the same time their product shall equal $-\frac{A_1}{3}$, equation (6), is as follows:

$$\left.\begin{array}{l} x = m + n, \text{ giving the root of equation (15)}, \\ \left.\begin{array}{l} x = am + a^2n, \\ x = a^2m + an, \end{array}\right\} \text{giving the roots of equation (17)}, \end{array}\right\} \quad (18)$$

The roots given by (17) may be simplified as follows:

Since $y = m$ and $z = n$, we have $yz = mn$. Comparing this with (6), we find $A_1 = -3mn$, this value of A_1, substituted in (17), gives

$$\left.\begin{array}{l} x = -\dfrac{m+n}{2} + \dfrac{m-n}{2}\sqrt{-3}, \\ x = -\dfrac{m+n}{2} - \dfrac{m-n}{2}\sqrt{-3}. \end{array}\right\} \quad (19)$$

Collecting in one point of view, the roots of the equation $x^3 + A_1x + A_2 = 0$, we have

$$\left.\begin{array}{ll} x = m + n, & (1) \\ x = -\dfrac{m+n}{2} + \dfrac{m-n}{2}\sqrt{-3}, & (2) \\ x = -\dfrac{m+n}{2} - \dfrac{m-n}{2}\sqrt{-3}, & (3) \end{array}\right\} \quad (B)$$

wnere m and n are given by equations (14).

We will now see what conditions must be fulfilled, in order that one or all of the roots may be real.

CASE I.

When $\frac{A_2^2}{4}+\frac{A_1^3}{27}$, or $\left(\frac{A_2}{2}\right)^2+\left(\frac{A_1}{3}\right)^3=0.$

In this case, the values of m and n are real, and each equal to $\sqrt[3]{-\frac{A_2}{2}}$, and the values of x, given by (B), are

$$\left.\begin{array}{lr} 2\sqrt[3]{-\frac{A_2}{2}}, & (1) \\ -\sqrt[3]{-\frac{A_2}{2}}, & (2) \\ -\sqrt[3]{-\frac{A_2}{2}}. & (3) \end{array}\right\} \quad (B')$$

CASE II.

When $\left(\frac{A_2}{2}\right)^2+\left(\frac{A_1}{3}\right)^3>0.$

In this case, the values of m and n are both real, and unequal. Hence, the first root as given by equation (B), is real, whilst the other two are imaginary.

CASE III.

When $\left(\frac{A_2}{2}\right)^2+\left(\frac{A_1}{3}\right)^3<0.$

In this case, since $\left(\frac{A_2}{2}\right)^2$ is positive for all values of A_2, it follows that $A_1<0$. This is called the ***irreducible case***, since m and n are both imaginary.

Nevertheless, we can prove, that in this irreducible case, all the roots are real. For,

$$\text{put} \quad \left.\begin{aligned}\left(\frac{A_2}{2}\right)^2+\left(\frac{A_1}{3}\right)^3&=-q^2,\\ -\frac{A_2}{2}&=p.\end{aligned}\right\} \qquad (C)$$

Then we shall have

$$\left.\begin{aligned}m&=(p+q\sqrt{-1})^{\frac{1}{3}},\\ n&=(p-q\sqrt{-1})^{\frac{1}{3}}.\end{aligned}\right\} \qquad (C)$$

See questions 13 and 14, Art. 191, which give the expanded form of m and n as follows :

$$(p+q\sqrt{-1})^{\frac{1}{3}}=$$
$$p^{\frac{1}{3}}+\frac{1}{3}p^{-\frac{2}{3}}q\sqrt{-1}+\frac{2}{3.6}p^{-\frac{5}{3}}q^2-\frac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3\sqrt{-1}-\&c.$$

$$(p-q\sqrt{-1})^{\frac{1}{3}}=$$
$$p^{\frac{1}{3}}-\frac{1}{3}p^{-\frac{2}{3}}q\sqrt{-1}+\frac{2}{3.6}p^{-\frac{5}{3}}q^2+\frac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3\sqrt{-1}-\&c.$$

Therefore, we find

$$\frac{m+n}{2}=p^{\frac{1}{3}}+\frac{2}{3.6}p^{-\frac{5}{3}}q^2-\&c.$$
$$\frac{m-n}{2}=\left\{\frac{1}{3}p^{-\frac{2}{3}}q-\frac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3+\&c.\right\}\sqrt{-1}.$$

Hence, the three values of x, as given by (B), become

$$\left.\begin{array}{l} x=2\left\{p^{\frac{1}{3}}+\dfrac{2}{3.6}p^{-\frac{5}{3}}q^2-\&c.\right\} \\ x=\left\{\begin{array}{l}-\left\{p^{\frac{1}{3}}+\dfrac{2}{3.6}p^{-\frac{5}{3}}q^2-\&c.\right\} \\ +\left\{\dfrac{1}{3}p^{-\frac{2}{3}}q-\dfrac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3+\&c.\right\}\sqrt{3},\end{array}\right\} \\ x=\left\{\begin{array}{l}-\left\{p^{\frac{1}{3}}+\dfrac{2}{3.6}p^{-\frac{5}{3}}q^2-\&c.\right\} \\ -\left\{\dfrac{1}{3}p^{-\frac{2}{3}}q-\dfrac{2.5}{3.6.9}p^{-\frac{8}{3}}q^3+\&c.\right\}\sqrt{3},\end{array}\right\} \end{array}\right\} \quad (B'')$$

where the values of p and q are given by (C).

And since p and q are real quantities, it follows that the three roots as given by (B″) are real.

GENERAL SOLUTION OF AN EQUATION OF THE FOURTH DEGREE.

(294.) Let the equation of the fourth degree be put under this form:

$$x^4+\mathcal{A}_1x^2+\mathcal{A}_2x+\mathcal{A}_3=0. \quad (1)$$

If we assume $$x=y+z+u, \quad (2)$$

we shall find $$x^2=y^2+z^2+u^2+2(yz+yu+zu),$$

or $$x^2-(y^2+z^2+u^2)=2(yz+yu+zu). \quad (3)$$

By squaring (3), we find

$$\left.\begin{array}{l} x^4-2(y^2+z^2+u^2)x^2+(y^2+z^2+u^2)^2= \\ 4(y^2z^2+y^2u^2+z^2u^2)+8yzu(y+z+u). \end{array}\right\} \quad (4)$$

Replacing $y+z+u$ by x, in (4), and transposing, we find

$$\left.\begin{array}{l} x^4-2(y^2+z^2+u^2)x^2-8yzu.x \\ +(y^2+z^2+u^2)^2-4(y^2z^2+y^2u^2+z^2u^2) \end{array}\right\}=0. \quad (5)$$

Now, in order that (5) and (1) may become identical, we must have

$$\left.\begin{array}{l} A_1 = -2(y^2+z^2+u^2), \\ A_2 = -8yzu, \\ A_3 = (y^2+z^2+u^2)^2 - 4(y^2z^2+y^2u^2+z^2u^2). \end{array}\right\} \quad (A)$$

From these conditions, we immediately deduce

$$\left.\begin{array}{l} y^2+z^2+u^2 = -\dfrac{A_1}{2}, \\[2ex] y^2z^2+y^2u^2+z^2u^2 = \dfrac{A_1^2-4A_3}{16}, \\[2ex] y^2z^2u^2 = \dfrac{A_2^2}{64}. \end{array}\right\} \quad (B)$$

Now, by Art. 265, we know that the sum of the three roots of a cubic equation with their signs changed, is equal to the coefficient of the second term of that equation; and the sum of their products taken two and two, is equal to the coefficient of the third term; also the continued product of the three roots is equal to the absolute term.

Hence, the values of y^2, z^2, and u^2, will correspond with the three roots of this equation:

$$t^3 + \frac{A_1}{2}.t^2 + \frac{A_1^2-4A_3}{16}.t - \frac{A_2^2}{64} = 0. \quad (6)$$

If we suppose $t = \frac{s}{4}$, equation (6) will become

$$s^3 + 2A_1s^2 + (A_1^2 - 4A_3)s - A_2^2 = 0.$$

If we denote the three roots of this equation, as found by method explained under Art. 293, by s', s'', s''', we shall have

$$\left.\begin{array}{l} y = \pm\frac{1}{2}\sqrt{s'}, \\ z = \pm\frac{1}{2}\sqrt{s''}, \\ u = \pm\frac{1}{2}\sqrt{s'''}. \end{array}\right\} \quad (C)$$

Now, in order to find x, a root of (1), we must add the values of y, z, and u, observing that their signs are so taken

that their continued product may be affected with a contrary sign with A_2, so as to satisfy the second condition of (A).

CASE I.

When $A_2 < 0$.

The four values will be as follows:

$$\left.\begin{array}{l} x = +\frac{1}{2}\sqrt{s'} + \frac{1}{2}\sqrt{s''} + \frac{1}{2}\sqrt{s'''}, \\ x = +\frac{1}{2}\sqrt{s'} - \frac{1}{2}\sqrt{s''} - \frac{1}{2}\sqrt{s'''}, \\ x = -\frac{1}{2}\sqrt{s'} + \frac{1}{2}\sqrt{s''} - \frac{1}{2}\sqrt{s'''}, \\ x = -\frac{1}{2}\sqrt{s'} - \frac{1}{2}\sqrt{s''} + \frac{1}{2}\sqrt{s'''}. \end{array}\right\} \quad \text{(D)}$$

CASE II.

When $A_2 > 0$.

The four values will be as follows:

$$\left.\begin{array}{l} x = -\frac{1}{2}\sqrt{s'} - \frac{1}{2}\sqrt{s''} - \frac{1}{2}\sqrt{s'''}, \\ x = -\frac{1}{2}\sqrt{s'} + \frac{1}{2}\sqrt{s''} + \frac{1}{2}\sqrt{s'''}, \\ x = +\frac{1}{2}\sqrt{s'} - \frac{1}{2}\sqrt{s''} + \frac{1}{2}\sqrt{s'''}, \\ x = +\frac{1}{2}\sqrt{s'} + \frac{1}{2}\sqrt{s''} - \frac{1}{2}\sqrt{s'''}. \end{array}\right\} \quad \text{(D}')$$

The method of solving a cubic equation as given under Art. 293, is generally supposed to have originated with *Cardan*, an Italian analyst of the 16th century; it is therefore frequently referred to as *Cardan's Method*. Montucla, in his *Histoire des Mathematiques*, seems to have proved that it was also discovered about the same time, independently of each other, by *Scipio Ferreus* and *Nicolas Tartalea*.

The above method for equations of the fourth degree, which is a close imitation of the method for cubic equations, was first given by *Euler*, a distinguished analyst.

As yet, analysts have not been able to obtain the general solution of equations beyond the fourth degree.

STURM'S THEOREM

(295.) Let $X=0$ be an algebraic equation having real coefficients; we will suppose, also, that it has no equal roots. Call X_1 its *first derived polynomial*, found by the method of Art. 273.

Apply to X and X_1 the method of finding the greatest common measure, as explained under Art. 50, with this condition, *always to change the sign of the remainder at each operation, and to use this remainder, thus modified, for a divisor in the next operation.*

Designate, moreover, by $X_2, X_3, X_4, \ldots\ldots X_r$, the successive remainders, taken with contrary signs.

If we denote the successive quotients by

$$q_1, q_2, q_3, \ldots\ldots q_{r-1},$$

we shall have the following relations:

$$\left.\begin{array}{lr} X=X_1q_1-X_2, & (1) \\ X_1=X_2q_2-X_3, & (2) \\ X_2=X_3q_3-X_4, & (3) \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ \ldots\ldots\ldots\ldots\ldots\ldots & \\ X_{r-2}=X_{r-1}q_{r-1}-X_r. & (r-1) \end{array}\right\} \quad (A)$$

We shall necessarily have X_r independent of x, and different from *zero*, (Art. 278.)

After having obtained the functions $X, X_1, X_2, \ldots\ldots X_r$, suppose we substitute in them for x, two numbers p and q of any signs whatever, p being $< q$.

The substitution of p will give results either positive or negative; if we only take account of the signs, and write them one after another in a line, they will give a certain number of variations and permanences.

The substitution of q will in like manner give a succession of signs, of a certain number of variations and permanences.

Now, the THEOREM OF STURM consists in this:

THE DIFFERENCE *between the number of variations given by the first series of signs, and the number of variations given by the second series of signs, will express exactly the* NUMBER *of real roots of the proposed equation, which are comprised between p and q.*

(296.) We shall now proceed to demonstrate this beautiful theorem.

I. Consider the function X in particular, and suppose a_1 is a real root of $X=0$. If we substitute a_1+u for x, in X, we shall obtain, Art. 273, a result of this form:

$$A+A'u+\frac{A''}{2}u^2+\frac{A''}{2.3}u^3\ldots\ldots u^n; \qquad (1)$$

where A is what X becomes when a_1 is put for x, and A', A'', A''',...... are the successive derived polynomials of A, found by the method of Art. 273.

Now, by hypothesis, a_1 is a root of $X=0$, therefore $A=0$, and the preceding expression becomes

$$u\left(A'+\frac{A''}{2}u+\frac{A'''}{2.3}u^2+\ldots\ldots+u^{n-1}\right). \qquad (2)$$

We can always take u sufficiently small to cause the quantity within the parenthesis of (2) *to have the same sign as its first term A'.*

II. *If, in the functions X, X_1, X_2,...... X_r, we substitute any quantity a for x, it cannot happen that two consecutive functions shall vanish at the same time.*

For take any three consecutive functions as X_{n-1}, X_n, X_{n+1}. Then, by conditions (A), we have

$$X_{n-1}=X_nq_n-X_{n+1}. \qquad (1)$$

Now, if we are able to have at the same time

$$X_{n-1}=0, \tag{2}$$

$$X_n=0, \tag{3}$$

we must also, by condition (1), have

$$X_{n+1}=0. \tag{4}$$

Since the relation (1) is general, it must be true when we write $n+1$ for n; hence we have

$$X_n=X_{n+1}q_{n+1}-X_{n+2}. \tag{5}$$

In (5), substituting the values of X_n, X_{n+1}, as given by (3) and (4), and we obtain

$$X_{n+2}=0. \tag{6}$$

By continuing this process, we should finally find

$$X_r=0, \tag{7}$$

which is absurd, since we have already shown that X_r cannot equal *zero*.

III. *The relation* $X_{n-1}=X_nq_n-X_{n+1}$, *shows that if a function* X_n *becomes* 0 *by the substitution of* $x=a$, *the two functions* X_{n-1}, X_{n+1}, *between which it is placed, have necessarily contrary signs for* $x=a$.

(297.) Designating by k a quantity positive or negative, but less than each of the the real roots of the equations,

$$\left.\begin{array}{r} X=0, \\ X_1=0, \\ X_2=0, \\ \cdots\cdots\cdots \\ \cdots\cdots\cdots \\ X_{r-1}=0, \end{array}\right\} \tag{B}$$

Conceive that the value of x is made to increase continuously from $x=k$, and that its successive values are substituted in the functions X, X_1, X_2, ... X_r. Now, so long as the increasing values of x are less than each of the roots of equations (B), the *signs*, arising from their substitution in

the functions of X, X_1, X_2, ... X_r, will occur in the same order; fòr, in order that the number of the variations and permanences of signs should change, it is necessary that some one of the above functions, as X_n, should have passed through the stage in which $X_n = 0$, which cannot have happened, since x is supposed less than the least value which can satisfy either of the equations (B).

(298.) We will now suppose that x has reached a value $x = a$, which causes one of the intermediate functions X_1, X_2, X_3, ... X_{r-1}, to vanish, without causing x to vanish. We will also suppose X_n to be the one which vanishes when $x = a$; then by II, under Art. 296, we know that X_{n-1}, X_{n+1}, cannot vanish, and by III, under the same Art., we also know that X_{n-1} and X_{n+1} must have contrary signs. Now, if we consider the sign of the vanishing term X_n to be either plus or minus, the three consecutive functions X_{n-1}, X_n, X_{n+1}, can produce only these two combinations of signs,

$$\left.\begin{array}{lc} & + \ \pm \ - \\ \text{or} & - \ \pm \ + \end{array}\right\} \qquad (8)$$

Either of which gives one variation and one permanence.

We know, by Art 297, that the signs of X_{n-1}, X_{n+1}, will not be changed from $x = k$ to $x = a$, and since we are able to take u as small as we please, it follows that they will not be changed from $x = a$ to $x = a + u$.

Hence, the hypothesis $x = a$, introduced in the series of functions X, X_1, X_2, ..., can produce neither a gain or loss in the number of variations.

(299.) We will now suppose that $x = a_1$ causes X to vanish. Let U and U_1 represent the values of X and X_1, when $x = a_1 + u$.

Represent, as in Art. 273, by A, A', A'', ..., the va-

lues of X and its successive derived functions, when $x=a$. In the same way, represent by A_1, A_1', A_1'',......, the values of X_1 and its successive derived functions.

By Art. 273, we shall have

$$\left.\begin{array}{l} U=A+A'u+\dfrac{A''}{2}u^2+\ldots\ldots \\ \\ U_1=A_1+A_1'u+\dfrac{A_1''}{2}u^2+\ldots\ldots \end{array}\right\} \quad (C)$$

Since a_1 is a root of $X=0$, we must have $A=0$. Again, the values A' and A_1 each represents the value of X_1 when a_1 is put for x, and since the equation $X=0$ is supposed not to have any equal roots, A' or its equal cannot vanish, therefore (C) becomes

$$\left.\begin{array}{l} U=A'u+\dfrac{A''}{2}u^2\ldots\ldots\ldots \\ \\ U_1=A'+A_1'u+\dfrac{A_1''}{2}u^2+\ldots\ldots \end{array}\right\} \quad (D)$$

. which the right-hand members will have the same signs as their first terms $A'u$, A', if we take u sufficiently small.

Hence, when u is positive, U and U_1 will have the same sign.

When u is negative, U and U_1 will have contrary signs.

From which it follows that the functions X and X_1 will give a variation for $x=a_1-u$, and a permanence for $x=a_1+u$.

Consequently, in the passage of the continuously increasing values of x from $x=a_1-u$ to $x=a_1+u$ a variation will be changed into a permanence.

The same results would have place, if the value $x=a_1$, which causes X to vanish, should at the same time cause some one or more of the functions X_1, X_2, X_3,........to vanish. (Art. 298).

Now, commencing with $x=a_1+u$, if we suppose the

value of x to increase continuously, the number of variations in the series of signs will remain the same, although the order of the succession of the signs may be changed until we reach another value, $x=a_2$, which causes X to vanish, and which is therefore a root of $X=0$; in which case a second variation must be changed into a permanence; and so on.

Hence, the number of variations lost when x increases from $x=k$ to $x=k'$, must be equal to the number of real roots of $X=0$, comprised between k and k'.

APPLICATION OF STURM'S THEOREM.

(300.) Before passing to the application of this theorem, we shall do well to pay attention to the following principles:

I. In obtaining the functions X, X_1, X_2,.... X_r, we are, by Art. 53, at liberty to introduce or suppress any numerical factor, provided that it is positive; but it is necessary to pay particular attention to the signs, and make only the changes mentioned under Art. 295, as the peculiarities of this theorem depend principally upon this change of the signs of X, X_1, X_2,.... X_r.

II. If we simply wish to know the total number of real roots, without fixing in any manner their limits, we need only substitute in the first terms of X, X_1, X_2,.... X_r, the values $-\infty$ and $+\infty$.

EXAMPLES.

1. How many real roots has the equation $8x^3-6x-1=0$?

The first derived of $8x^3-6x-1$ is $24x^2-6$, or suppressing the positive numerical factor 6, it becomes $4x^2-1$.

Now, applying to $8x^3-6x-1$ and $4x^2-1$ the method of finding the greatest common divisor, we obtain $-4x-1$ for the first remainder, changing its signs it becomes $4x+1$,

continuing the operation with $4x^2-1$ and $4x+1$, we find -3 for the remainder, hence we have

$$\left.\begin{aligned} X &= 8x^3-6x-1, \\ X_1 &= 4x^2-1, \\ X_2 &= 4x+1, \\ X_3 &= 3. \end{aligned}\right\} \quad \text{(A)}$$

Now, if for x in the above functions we substitute $-\infty$, the signs of the results will be $-+-+$ giving 3 variations.

If we substitute $+\infty$, they will be $++++$ giving 0 variations.

Hence, the number of real roots is $3-0=3$.

If in the same functions (A), we substitute the three consecutive values $x=-1$, $x=0$, $x=1$, we shall find that

for $x=-1$	the signs are	$-+-+$	giving	3 variations,
" $x=0$	"	$--++$	"	1 "
" $x=1$	"	$++++$	"	0 "

Hence, two roots lie between -1 and 0; and one root between 0 and 1.

If we substitute $x=-\frac{1}{2}$, we shall find $+\pm-+$ giving 2 variations.

Therefore, one of the negative roots lies between -1 and $-\frac{1}{2}$, and the other between $-\frac{1}{2}$ and 0.

2. How many real roots has $x^3-5x^2+8x-1=0$?

In this example, we find

$$\begin{aligned} X &= x^3-5x^2+8x-1, \\ X_1 &= 3x^2-10x+8, \\ X^2 &= 2x-31, \\ X_3 &= -2295. \end{aligned}$$

When $x=-\infty$,	we find	$-+--$,	giving	2 variations,
" $x=+\infty$,	"	$+++-$,	"	1 "

Therefore, the above equation has but one real root, and consequently, it must have two imaginary roots.

3. How many real roots has $x^4-2x^3-7x^2+10x+10=0$?

In this example, we find

$$X=x^4-2x^3-7x^2+10x+10,$$
$$X_1=2x^3-3x^2-7x+5,$$
$$X_2=17x^2-23x-45,$$
$$X_3=152x-305,$$
$$X_4=524785.$$

When $x=-\infty$, we find $+-+-+$, giving 4 variations.
" $x=+\infty$, " $+++++$, " 1 "

Consequently, the roots are all real.

We also find

when $x=$	0	$+\ +\ -\ -\ +$	giving	2	variations,
" $x=$	1	$+\ -\ -\ -\ +$	"	2	"
" $x=$	2	$+\ -\ -\ -\ +$	"	2	"
" $x=$	3	$+\ +\ +\ +\ +$	"	0	"
" $x=$	-1	$-\ +\ +\ -\ +$	"	3	"
" $x=$	-2	$-\ -\ +\ -\ +$	"	3	"
" $x=$	-3	$+\ -\ +\ -\ +$	"	4	"

From which we see that the equation has two positive roots between 2 and 3; one negative root between 0 and -1; and one negative root between -2 and -3.

4. How many real roots has $2x^4-13x^2+10x-19=0$?
Here we find

$$X=2x^4-13x^2+10x-19,$$
$$X_1=4x^3-13x+5,$$
$$X_2=13x^2-15x+38.$$

It is not necessary to calculate X_3 and X_4, since the two roots of $X_2=13x^2-15x+38=0$, are imaginary, for $(15)^2<4\times13\times38$. See Art. 149, Formula (B).

Using only the values X, X_1, X_2, we have

when $x = -\infty$ $\quad + \; - \; +$ giving 2 variations,
" $x = +\infty$ $\quad + \; + \; +$ " 0 "

Therefore, the two remaining roots are real.

5. How many real roots has $x^5 - 36x^3 + 72x^2 - 37x + 72 = 0$?

Here we find

$$X = x^5 - 36x^3 + 72x^2 - 37x + 72,$$
$$X_1 = 5x^4 - 108x^2 + 144x - 37,$$
$$X_2 = 18x^3 - 54x^2 + 37x - 90,$$
$$X_3 = 1319x^2 - 2487x - 684,$$
$$X_4 = -2960933x + 34935426,$$
$$X_5 = -,$$

when $x = -\infty$ we have $- + - + + -$ giving 4 variations,
" $x = +\infty$ " $+ + + + - -$ " 1 "

Hence, the proposed equation has three real roots and two imaginary ones.

6. How many real roots has $x^3 + A_1x + A_2 = 0$?

In this example, we find

$$X = x^3 + A_1x + A_2,$$
$$X_1 = 3x^2 + A_1,$$
$$X_2 = -2A_1x - 3A_2,$$
$$X_3 = -4A_1^3 - 27A_2^2.$$

CASE I.

When $-4A_1^3 - 27A_2^2 > 0.$

Now, since $-27A_2^2$ is negative for all values of A_2, it is necessary that $A_1 < 0$, in order to fulfil the above condition. Consequently,

when $x = -\infty$ we have $- + - +$ giving 3 variations,
" $x = +\infty$ " $+ + + +$ " 0 "

Therefore, when $-4A_1^3-27A_2^2>0$ or $4A_1^3+27A_2^2<0$, then the three roots are real. See Case III, page 370.

CASE II.

When $-4A_1^3-27A_2^2<0$.

This condition can be fulfilled for values of A_1 either positive or negative, so that

when $x=-\infty$ we have $-+\pm-$ giving 2 variations.
" $x=+\infty$ " $++\mp-$ " 1 "

Therefore, when $-4A_1^3-27A_2^2<0$, or $4A_1^3+27A_2^2>0$, then there will be but one real root, and consequently two imaginary roots. See Case II, page 370.

CASE III.

When $-4A_1^3-27A_2^2=0$.

In this case, we know, by Art. 279, that there are two equal roots which will be given by $2A_1x+3A_2=0$. Hence, one of the equal roots is $x=-\frac{3A_2}{2A_1}$, and the other root must be $x=\frac{3A_2}{A_1}$. See Case I, page 370.

7. How many real roots has $x^4+A_1x^2+A_2x+A_3=0$?

Here we find

$$X=x^4+A_1x^2+A_2x+A_3,$$
$$X_1=4x^3+2A_1x+A_2,$$
$$X_2=-2A_1x^2-3A_2x-4A_3,$$
$$X_3=(8A_1A_3-2A_1^3-9A_2^2)x-A_2(A_1^2+12A_3),$$
$$X_4=16A_3(A_1^2-4A_3)^2-A_2^2(4A_1^3-144A_1A_3+27A_2^2)$$

CASE I.

When $A_1 < 0$, $8A_1A_3 - 2A_1^3 - 9A_2^2 > 0$, *and*
$16A_3(A_1^2 - 4A_3)^2 > A_2^2(4A_1^3 - 144A_1A_3 + 27A_2^2)$.

Then,

when $x = -\infty$, we find $+-+-+$, giving 4 variations.
" $x = +\infty$, " $+++++$, " 0 "

Therefore, in this case all the roots are real.

CASE II.

When $16A_3(A_1^2 - 4A_3)^2 < A_2^2(4A_1^3 - 144A_1A_3 + 27A_2^2)$.

Then,

when $x = -\infty$, we find $+--+-$, giving 3 variations.
" $x = +\infty$, " $++---$, " 1 "

Therefore, in this case, there must be two real roots, and consequently, two imaginary roots.

When neither of these conditions are fulfilled, all the roots are imaginary.

GENERAL METHOD OF ELIMINATION AMONG EQUATIONS ABOVE THE FIRST DEGREE.

(301.) Suppose we have two equations, each containing x and y, represented by

$$X = 0, \qquad (1)$$
$$X_1 = 0, \qquad (2)$$

Now, if we seek the greatest common measure of the polynomials X and X_1, by the method of Art. 50, we shall have

$$X = X_1q + r, \qquad (3)$$

where q is the quotient of X divided by X_1, and r is the remainder. Now, since by (1) and (2), X and X_1 are each zero, it follows that r as given by (3), must also be zero.

(302.) *From which we conclude, that if we operate upon the polynomials X and X_1, by the method for finding the greatest common measure, we shall have the successive remainders each equal to zero.*

If we arrange the polynomials with reference to either of the letters, before operating upon them, we shall ultimately find a remainder independent of that letter, when the polynomials have no common measure, which remainder being put equal to zero, will give an equation containing but one unknown quantity.

When the two polynomials have a common measure, it must be put equal to zero, if it contains both the unknown quantities, then divide both polynomials by it, and proceed with the results as in the first case.

Note.—In the operation of finding the greatest common measure of two polynomials, it frequently becomes necessary to suppress factors, as well as to introduce new factors. When this is done, we must carefully examine whether such factors are able to effect the final result. If no factors other than numerical, are either suppressed or introduced, then the above method is rigidly correct, but in other cases, the rule would require some modification.

EXAMPLES.

1. Obtain from the two equations

$$x^2+xy+y^2-1=0, \qquad (1)$$
$$x^3+y^3=0, \qquad (2)$$

a single equation in terms of y.

Proceeding by the method of finding the greatest common measure, Art. 50, we have for the

FIRST OPERATION.

$$\begin{array}{l|l}
\quad x^3+y^3 & x^2+yx+y^2-1 \\
\quad \underline{x^3+yx^2+(y^2-1)x} & \overline{x-y.} \\
-yx^2-(y^2-1)x+y^3 & \\
\underline{-yx^2-y^2x-y^3+y} & \\
\qquad x+2y^3-y = \text{first remainder.} &
\end{array}$$

Again, dividing x_2+yx+y^2-1 by this remainder, we find for the

SECOND OPERATION.

$$\begin{array}{l|l}
x^2+yx+y^2-1 & x+2y^3-y \\
\underline{x^2+(2y^3-y)x} & \overline{x-(2y^3-2y).} \\
-(2y^3-2y)x+y^2-1 & \\
\underline{-(2y^3-2y)x-4y^6+6y^4-2y^2} & \\
\qquad 4y^6-6y^4+3y^2-1 = \text{second rem.} &
\end{array}$$

Putting this remainder, which is independent of x, equal to *zero*, we have for the equation sought:

$$4y^6-6y^4+3y^2-1=0. \qquad (3)$$

If we were required to find, from the above two equations, one single equation in terms of x, we observe, that all that would be necessary would be to change y into x, in equation (3), since x and y can be changed the one for the other in equations (1) and (2) without affecting their form.

2. Obtain an equation independent of y from the two equations

$$\left.\begin{array}{r}(y+1)x^6-(3y^3+3y^2-y-1)x^5+(2y^2-2)x^4\\+(3y^4+3y^3+3y+3)x^3-(2y^3-4y^2-4y-1)x^2\\-(2y^2+3y-4)x+y-1\end{array}\right\}=0, \quad (1)$$

$$\left.\begin{array}{r}3x^3y^3-(3x^5+2x^2)y^2+(2x^4+3x^2-2x)y\\+x^6+x^5-2x^4+3x^3+x^2+x+1\end{array}\right\}=0. \quad (2)$$

Proceeding with these equations agreeably to the above method, we find for the first remainder the following:

$$3x^2y^2-2xy+3x-2.$$

Repeating the process, we find for the second remainder, the followin g:

$$x^6+x^5+x^4+x^3+x^2+x+1,$$

which being put equal to zero, gives for the equation sought,

$$x^6+x^5+x^4+x^3+x^2+x+1=0.$$

3. Obtain an equation independent of y from the two equations

$$\left.\begin{array}{r}3x^2y^4+(3x^2-3x)y^3-(2x^2-x)y^2\\+(x^3-2x^2+2x-3)y+x^3-x-2\end{array}\right\}=0. \quad (1)$$

$$3x^2y^3-3xy^2-(2x^2-x)y+x^2+x-3=0. \quad (2)$$

Ans. $x^3-x-2=0.$

(303.) When we have three equations involving three unknown quantities, we must first eliminate one of the unknowns by combining either of the equations with the other two; we shall thus obtain two new equations involving only two unknown quantities, which, as we have just shown, will give a final equation involving but one unknown quantity.

EXAMPLES.

1. Obtain an equation containing only x, from the three equations

$$x+y^2-a=0. \qquad (1)$$
$$y+z^2-b=0. \qquad (2)$$
$$z+x^2-c=0. \qquad (3)$$

Eliminating z between equations (2) and (3) we have the following

OPERATION.

$$\begin{array}{l|l} z^2+y-b & z+x^2-c \\ z^2+(x^2-c)z & \overline{z-(x^2-c).} \\ \hline -(x^2-c)z+y-b & \\ -(x^2-c)z-x^4+2cx^2-c^2 & \\ \hline \quad x^4-2cx^2+y+c^2-b=\text{remainder.} & \end{array}$$

Putting this remainder equal to zero, we have

$$x^4-2cx^2+y+c^2-b=0. \qquad (4)$$

Now, eliminating y between equations (1) and (4), we have the following

OPERATION.

$$\begin{array}{l|l} y^2+x-a & y+x^4-2cx^2+c^2-b \\ y^2+(x^4-2cx^2+c^2-b)y & \overline{y-(x^4-2cx^2+c^2-b).} \\ \hline -(x^4-2cx^2+c^2-b)y+x-a & \\ -(x^4-2cx^2+c^2-b)y-(x^4-2cx^2+c^2-b)^2 & \\ \hline \quad (x^4-2cx^2+c^2-b)^2+x-a=\text{remainder.} & \end{array}$$

Expanding this remainder, and then equating it with zero, we have

$$\left.\begin{array}{r} x^8-4cx^6+(6c^2-2b)x^4-(4c^3-4bc)x^2 \\ +x+c^4+b^2-2bc^2-a \end{array}\right\}=0. \qquad (5)$$

By simply permutating these quantities, (Art. 85), we have

$$\left.\begin{array}{r} y^8-4ay^6+(6a^2-2c)y^4-(4a^3-4ca)y^2 \\ +y+a^4+c^2-2ca^2-b \end{array}\right\}=0, \qquad (6)$$

$$\left.\begin{array}{r}z^8-4bz^6+(6b^2-2a)z^4-(4ba^2-4ab)z^2\\+z+b^4+a^2-2ab^2-c\end{array}\right\}=0. \quad (7)$$

2. In a similar manner, find three equations each containing but one unknown quantity, from the three equations

$$x^2+xy-a=0, \quad (1)$$
$$y^2+yz-b=0, \quad (2)$$
$$z^2+zx-c=0. \quad (3)$$

First, eliminating z between (2) and (3), we find

$$y^4-xy^3-(2b+c)y^2+bxy+b^2=0. \quad (4)$$

Secondly, eliminating y between (1) and (4), we find

$$\left.\begin{array}{r}2x^8-(7a+3b+c)x^6+(9a^2+5ab+2ac+b^2)x^4\\-(5a^3+2a^2b+a^2c)x^2+a^4\end{array}\right\}=0. \quad (5)$$

By permutating these letters, we find

$$\left.\begin{array}{r}2y^8-(7b+3c+a)y^6+(9b^2+5bc+2ba+c^2)y^4\\-(5b^3+2b^2c+b^2a)y^2+b^4\end{array}\right\}=0, \quad (6)$$

$$\left.\begin{array}{r}2z^8-(7c+3a+b)z^6+(9c^2+5ca+2cb+a^2)z^4\\-(5c^3+2c^2a+c^2b)z^2+c^4\end{array}\right\}=0. \quad (7)$$

If $a=16$, $b=17$, and $c=18$.

Then will the eight sets of values be

$$1.\left\{\begin{array}{l}x=\pm 4.173281,\\y=\pm 4.287098,\\z=\mp 0.330363.\end{array}\right.$$

$$2.\left\{\begin{array}{l}x=\pm 2.525516,\\y=\pm 2.969156,\\z=\pm 3.240579.\end{array}\right.$$

$$3.\left\{\begin{array}{l}x=\pm 0.418924,\\y=\pm 3.912240,\\z=\pm 4.048877.\end{array}\right.$$

$$4.\left\{\begin{array}{l}x=\mp 4.003756,\\y=\pm 0.007100,\\z=\mp 4.245971.\end{array}\right.$$

3. Find three equations each containing but one unknown quantity, from the three equations

$$x + yz - a = 0, \qquad (1)$$
$$x + zx - b = 0, \qquad (2)$$
$$z + xy - c = 0. \qquad (3)$$

Operating as in the preceding examples, we find the following results:

$$x^5 - ax^4 - 2x^3 + (2a + bc)\,x^2 - (b^2 + c^2 - 1)x + bc - a = 0,$$
$$y^5 - by^4 - 2y^3 + (2b + ca)\,y^2 - (c^2 + a^2 - 1)y + ca - b = 0,$$
$$z^5 - cz^4 - 2z^3 + (2c + ab)\,z^2 - (a^2 + b^2 - 1)z + ab - c = 0.$$

4. Find three equations each containing but one unknown quantity, from the three equations

$$x^2 + yz - a = 0, \qquad (1)$$
$$y^2 + zx - b = 0, \qquad (2)$$
$$z^2 + xy - c = 0. \qquad (3)$$

$$\text{Ans.} \left\{ \begin{array}{l} \left.\begin{array}{l} 8x^8 - 20ax^6 + (18a^2 - 2bc)\,x^4 \\ \quad + (5abc - 7a^3 - b^3 - c^3)x^2 + (a^2 - bc)^2 \end{array}\right\} = 0, \\ \left.\begin{array}{l} 8y^8 - 20by^6 + (18b^2 + 2ca)\,y^4 \\ \quad + (5bca - 7b^3 - c^3 - a^3)y^2 + (b^2 - ca)^2 \end{array}\right\} = 0, \\ \left.\begin{array}{l} 8z^8 - 20cz^6 + (18c^2 - 2ab)\,z^4 \\ \quad + (5cab - 7c^3 - a^3 - b^3)z^2 + (c^2 - ab)^2 \end{array}\right\} = 0. \end{array} \right.$$

(304.) When there are four equations, we must first reduce the number to three by eliminating any one of the unknown quantities, and then proceed as above. From what has already been done, it will not be difficult to know how to proceed for a greater number of equations, but it is obvious that in many cases this general method must be very long and laborious, still it is valuable on account of the certainty of the result.

CHAPTER XI.

NUMERICAL SOLUTION OF CUBIC EQUATIONS, AND EQUATIONS OF SUPERIOR DEGREES.

(305.) Let $$A_1x^3+A_2x^2+A_3x=A_4, \qquad (1)$$
be any cubic equation, and suppose that two consecutive numbers in either of the series

$$\left.\begin{array}{r} 1,\ 2,\ 3,\ 4,\ \&c. \\ 10,\ 20,\ 30,\ 40,\ \&c. \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ 0.1,\ 0.2,\ 0.3,\ 0.4,\ \&c. \\ 0.01,\ 0.02,\ 0.03,\ 0.04,\ \&c. \\ \&c. \qquad\qquad \&c. \end{array}\right\} \quad (A)$$

are found such, that by substituting the first for x in equation (1), the result shall be less than A_4, and by substituting the second, the result shall be greater than A_4; then, by Art. 263, the first of these numbers, omitting the cyphers if it have any, will be the first figure of one of the roots. Let this figure be denoted by r_1, and the other succeeding figures of the same root by r_2, r_3, r_4, &c., respectively. That is, r_1, r_2, r_3, r_4, &c., are the local values of the successive figures of the root.

If for x, in equation (1), we substitute its first figure r_1, we shall have $$A_1r_1^3+A_2r_1^2+A_3r_1=A_4. \qquad (2)$$

Therefore, $$r_1=\frac{A_4}{A_3+A_2r_1+A_1r_1^2}. \qquad (3)$$

If we put y for the excess of the true root above its first figure, we shall have $x = r_1 + y$; this value being substituted in (1), we get

$$\begin{array}{rcl} A_3 y + A_3 r_1 & = & A_3 x, \\ A_2 y^2 + 2A_2 r_1 y + A_2 r_1^2 & = & A_2 x^2, \\ A_1 y^3 + 3A_1 r_1 y^2 + 3A_1 r_1^2 y + A_1 r_1^3 & = & A_1 x^3, \\ \hline A_1 y^3 + \ A'_2 y^2 \ + \ A'_3 y \ + \ B & = & A_4, \end{array}$$

or
$$A_1 y^3 + A'_2 y^2 + A'_3 y = A'_4, \qquad (4)$$

where
$$\left.\begin{array}{ll} A'_2 = A_2 + 3A_1 r_1, & (1) \\ A'_3 = A_3 + 2A_2 r_1 + 3A_1 r_1^2, & (2) \\ A''_4 = A_4 - A_3 r_1 - A_2 r_1^2 - A_1 r_1^3. & (3) \end{array}\right\} \quad (B)$$

Equation (4) is in all respects similar to the original equation (1); therefore, repeating the above process upon this equation, we shall obtain

$$r_2 = \frac{A'_4}{A'_3 + A'_2 r + A_1 r_2^2}, \qquad (5)$$

where r_2 is the first figure of the root of equation (4), or the second figure of the root of equation (1). Putting z for the sum of all the remaining figures, we have $y = r_2 + z$; this value substituted in (4), gives

$$\begin{array}{rcl} A'_3 z + A'_3 r_2 & = & A'_3 y, \\ A'_2 z^2 + 2A'_2 r_2 z + A'_2 r_2^2 & = & A'_2 y^2, \\ A_1 z^3 + 3A_1 r_2 z^2 + 3A_1 r_2^2 z + A_1 r_2^3 & = & A_1 y^3, \\ \hline A_1 z^3 + \ A''_2 z^2 \ + \ A''_3 z + \ B' & = & A'_4, \end{array}$$

or
$$A_1 z^3 + A''_2 z^2 + A''_3 z = A''_4, \qquad (6)$$

where
$$\left.\begin{array}{ll} A''_2 = A'_2 + 3A_1 r_2, & (1) \\ A''_3 = A'_3 + 2A'_2 r_2 + 3A_1 r_2^2, & (2) \\ A_4'' = A'_4 - A'_3 r_2 - A'_2 r_2^2 - A_1 r_2^3. & (3) \end{array}\right\} \quad (B')$$

Here, again, equation (6) is similar to equations (4) and (1)

We might now proceed to find the first figure of the root of equation (6), the value of which must be such, that we shall have

$$r_3 = \frac{A''_4}{A''_3 + A''_2 r_3 + A_1 r_3^2}. \qquad (7)$$

(306.) Now, by observing the formation of the coefficients A'_3, A'_2, in equation (5), and recollecting that r_1 being the first figure of the root, must be greater than r_2, it will appear obvious that A'_3 must constitute the largest portion of $A'_3 + A'_2 r_2 + A_1 r_2^2$, which is the denominator of the value r_2 as given by (5), and if r_1 is already known, then (2), of (B), will make known A'_3, which may be used as a *trial divisor* for finding r_2, the second figure of the root; the same may be observed of the succeeding divisors, and it is obvious that these trial divisors A''_3, A'''_3, &c., will continually approach nearer the true divisors.

(307.) If we multiply the first coefficient by r_1, and add the product to the second coefficient, we shall find

$$A_2 + A_1 r_1. \qquad (8)$$

Multiplying expression (8) by r_1, and adding the product to the third coefficient, we have

$$A_3 + A_2 r_1 + A_1 r_1^2. \qquad (9)$$

Multiplying expression (9) by r_1, and subtracting the product from A_4, we have

$$A_4 - A_3 r_1 - A_2 r_1^2 - A_1 r_1^3. \qquad (10)$$

Again, multiplying the first coefficient by r_1, and adding the product to expression (8), we have

$$A_2 + 2A_1 r_1. \qquad (11)$$

Multiplying expression (11) by r_1, and adding the product to expression (9), we have

$$A_3 + 2A_2 r_1 + 3A_1 r_1^2. \qquad (12)$$

Again, multiplying the first coefficient by r_1, and adding the product to expression (11), we have

$$A_2 + 3A_1r_1. \tag{13}$$

Expressions (13), (12), and (10), are the values of the coefficients A'_2, A'_3 and A'_4 respectively, of equation (4), as given by (B).

(308.) From the above method of operation, we discover that the root of a cubic equation having numerical coefficients, can be found by the following

RULE.

I. *Having found the first figure of the root, multiply it into the first coefficient, and add the product to the second coefficient, which sum, multiply by the same figure and add the product to the third coefficient, multiply this last sum by the same figure and subtract the product from the term which constitutes the right-hand member of the equation; the remainder we shall call the* FIRST DIVIDEND.

Again, multiply the first coefficient by the same figure, and add the product to the last number under the second coefficient, which sum must be multiplied by the same figure and the product added to the last number under the third coefficient, the result we shall call the FIRST TRIAL DIVISOR.

Again, multiply the first coefficient by this same figure, and add the product to the last number under the second coefficient.

II. *Find the second figure of the root by dividing the* FIRST DIVIDEND *by the* FIRST TRIAL DIVISOR, *proceed with this second figure precisely as was done with the first figure, observing to keep the work so that units shall stand under units, tens under tens, &c., &c.*

NOTE.—The above rule bears a close resemblance to the rule for extracting the cube root of a polynomial, as given under Art. 106.

EXAMPLES.

1. Find a root of the cubic equation $3x^3+2x^2+4x=75$.

OPERATION.

3 = 1st coefficient.	2 = 2d coefficient.	4 = 3d coefficient.	75 = absolute term. (2 = 1st figure. 5 = 2d figure. 7 = 3d figure. 7 = 4th figure. 9 = 5th figure. &c.)
3	2	4	75(2.5 7 7 9 &c. $=x$.
	8	20	40
	14	48 = 1st trial divisor.	—
	20	5875	35 = first dividend.
	215	7025 = 2d trial divisor.	29375
	230	719797	——
	245	737241 = 3d trial divisor.	5625 = 2d dividend.
	2471	73900157	5038579
	2492	74076361 = 4th trial divisor.	——
	2513	7409903713	586421 = 3d dividend.
	25151		517301099
	25172		——
	25193		69119901 = 4th dividend.
	251957		66689133417
			2430767583.

2. Find one of the roots of the equation

$$7x^3+x^2-14x=1675.$$

7	1	-14	1675(6.2676&c. $=x$.
	43	244	1464
	85	754	——
	127	77968	211
	1284	80564	155936
	1298	8135372	——
	1312	8214596	55064
	13162	822387163	48812232
	13204	823315069	——
	13246	82339463572	6251768
	132509		5756710141
	132558		——
	132607		495057859
	1326112		494036781432
			1021077568.

3. Find a root of the equation $3x^3-2x^2+x=3$.

3	-2	1	3(1.1417&c.$=x$.
	1	2	2
	4	6	—
	7	673	1
	73	749	673
	76	78108	—
	79	81364	327
	802	8144663	312432
	814	8152929	—
	826	815871877	14568
	8263		8144663
	8266		—
	8269		6423337
	82711		5711103139
			—
			712233861.

(309.) The above roots are all positive. We will now give a couple of examples when the value of x is negative. The operation will remain the same if we observe the algebraic rule for the signs.

4. Find a root of the equation $5x^3-6x^2+3x=-85$.

5	-6	3	-85(-2.16139&c.$=x$
	-16	35	-70
	-26	87	—
	-36	9065	-15
	-365	9435	-9065
	-370	966180	—
	-375	989040	-5935
	-3780	98942405	-5797080
	-3810	98980815	—
	-3840	9899233995	-137920
	-38405	9900386535	-98942405
	-38410	990073231455	—
	-38415		-38977595
	-384165		-29697701985
	-384180		—
	-384195		-9279893015
	-3841995		-8910659083095
			—
			-369233931905.

5. Find a root of the equation $x^3+x^2+70x=-300$.

1	1	70	—300(—3.73879&c. = x.
	—2	76	—228
	—5	91	———
	—8	9709	—72
	—87	10367	—67963
	—94	1039739	———
	—101	1042787	—4037
	—1013	104360284	—3119217
	—1016	104441932	———
	—1019	10444908229	—917783
	—10198	10445623307	—834882272
	—10206	1044571525271	———
	—10214		—82900728
	—102147		—73114357603
	—102154		———
	—102161		—9786370397
	—1021619		—9401143727439
			———
			—385226669561.

(310.) When the second or first power of x is wanting, we must consider its coefficient as being ± 0.

6. Find a root of the equation $x^3-12x=28$.

1	± 0	—12	28(4.30213&c. = x.
	4	4	16
	8	36	———
	12	3969	12
	123	4347	11907
	126	43495804	———
	129	43521612	93
	12902	4352290261	86991608
	12904	4352419323	———
	12906	435245804199	6008392
	129061		4352290261
	129062		———
	129063		1656101739
	1290633		1305737412597
			———
			350364326403.

7. Find a root of the equation $2x^3+3x^2=850$.

2	3	± 0	850(7.05025&c. $=x$.
	17	119	833
	31	336	—
	45	3382550	17
	4510	3405150	16912750
	4520	34052406008	—
	4530	34053312024	87250
	453004	3405353853050	68104812016
	453008		—
	453012		19145187984
	4530130		17026769265250
			2118418718750.

(311.) In all the preceding examples, the first figure of the root has been in the units' place; we will now add two examples in which the first figure is in the tens' place.

8. Find a root of the equation $3x^3-7x^2+13x=45000$.

3	-7	13	45000(25.404&c. $=x$.
	53	1073	21460
	118	3333	—
	173	4278	23540
	188	5288	21365
	203	587568	—
	218	546384	2175
	2192	5464726448	2150272
	2204		—
	2216		24728
	221612		21858905792
			2869094208.

9. Find a root of the equation $x^3+60x^2-800x=60000$

1	60	-800	60000(30.537&c. $=x$
	90	1900	57000
	120	5500	—
	150	557525	3000
	1505	565075	2787625
	1510	56552959	—
	1515	56598427	212375
	15153	5660903879	169658877
	15156		—
	15159		42716123
	151597		39626327163
			3089795847.

(312.) The method of proceeding, when the first figure is of a local value greater than ten, will be obvious.

We will add two examples in which the first figure is in the tenth's place.

10. Find a root of $10x^3-24x^2-30x=-6$.

10	-24	-30	-6 (0.1768 &c.$=x$.
	-23	-323	-323
	-22	-345	
	-21	-35921	-277
	-203	-37293	-251447
	-196	-3740604	
	-189	-3751872	-25553
	-1884	-275336896	-22443624
	-1878		
	-1872		-3109376
	-18712		-3002695168
			$-106680832.$

11. Find a root of the equation $x^3+9x=6$.

1	± 0	9	6 (0.6378 &c.$=x$.
	0.6	936	5616
	12	1008	
	18	101349	384
	183	101907	304047
	186	10203979	
	189	10217307	79953
	1897	1021883644	71427853
	1904		
	1911		8525147
	19118		8175069152
			350077848.

(313.) By reviewing the foregoing examples, we discover that the last terms under the third coefficient, or the *trial divisors*, remain unchanged in several of its left-hand figures, thus in example 1, 740 is common to the left-hand of the last two terms under the third coefficient. In example 2, the figures common are 8233. In example 3, the figures common, are 815, and so for the succeeding examples.

It is evident, that if we omit all on the right of the constant figures of the last term under the third coefficient, and also omit from the right of the last dividend, the same number of figures save one, we may then divide the remaining figures of the dividend by the remaining figures of the last term under the third coefficient, by long division; and as many additional figures of the root may in this way be found as there are figures in the divisor thus used.

12. Find a root, to 8 decimals, of the equation

$$x^3+x^2=500.$$

1	1	± 0	500(7.61727975&c.$=x$.
	8	56	392
	15	161	
	22	17456	108
	226	18848	104736
	232	1887181	
	238	1889563	3264
	2381	189123159	1887181
	2382	189290067	
	2383	18929483724	1376819
	23837	18929.960752	1323862113
	23844		
	23851		52956887
	238512		37858967448
	258514		
			150979.19552
			132509
			18470
			17036
			1434
			1325
			109
			95
			14.

EXPLANATION.

After finding 4 decimal places in the root by the preceding rule, we cut off 6 figures from the right of the last trial divisor, thus leaving the constant figures 18929; and from the right of the dividend we cut off 5 figures, leaving 150979; we then divided 150979 by 18929, by the rule for abridged division, (see Higher Arithmetic,) and thus obtained the additional figures of the root.

13. Find a root, to 10 decimals, of the equation

$$x^3 - 17x^2 + 54x = 350.$$

1	-17	54	350)14.9540686096.
	-7	-16	-160
	3	14	——
	13	82	510
	17	166	328
	21	18931	——
	25	21343	182
	259	2148175	70379
	268	2162075	——
	277	2163189	11621
	2775	216430348	10740875
	2780	2164320197236	——
	2785		880125
	27854		865275664
	27858		——
	27862		14849336
	2786206		12985921183416
			——
			186341.4816584
			173147
			——
			13194
			12986
			——
			208
			194
			——
			14
			13
			——
			1.

314.) Thus far we have sought only one of the roots of our equations. If we wish the three roots, we may divide the given equation, when all the terms are transposed to one side, by the unknown quantity *minus* the value of the root found by the above method, we shall thus depress the cubic equation to a quadratic. (See Art. 255.)

14. Find the three roots of the equation

$$x^3-15x^2+63x=50.$$

Here, we soon discover that one of the roots lies between 1 and 2; seeking this root by the above process, we find

1	—15	63	50(1.028039231 &c.$=x$.
	—14	49	49
	—13	36	—
	—12	357604	1
	—1198	355212.	715208
	—1196	35425744	———
	—1194	35330352	284792
	—11932	353299945209.	283405952
	—11924	35329.6370427	———
	—11916		1386048
	—1191597		1059899835627
	—1191594		———
			326148.164373
			317967
			———
			8181
			7066
			———
			1115
			1060
			———
			55
			35
			—
			20.

Now, dividing $x^3-15x^2+63x-50$ by $x-1.02803923$ we find, for a quotient, the following:

$$x^2 - 13.97196077x + 48.6362762.$$

Hence, we have this quadratic equation,

$$x^2 - 13.97196077x = -48.6362762.$$

This solved by the usual rule for quadratics, gives the following values:

$$x = 6.576535\ ;\ x = 7.395426.$$

Therefore, the three roots of $x^3 - 15x^2 + 63x = 50$, are

$$1.028039\ ;\ 6.576535\ ;\ 7.395426.$$

(315.) From the work of the last example, we see that we need only seek one of the roots of a cubic equation by the foregoing rule, as the other two may then be found by the solution of a quadratic. When all the three roots are real, it will frequently be as simple to find them by the foregoing general method. But when two of the roots are imaginary, we must proceed agreeably to the last Art.

15. Find the three roots of the equation $x^3 - 15x = -21$.

Applying the principle of Sturm's Theorem, we find

$$\begin{aligned} X &= x^3 - 15x + 21, \\ X_1 &= x^2 - 5, \\ X_2 &= 10x - 21, \\ X_3 &= 59. \end{aligned}$$

For $x = -\infty$, we find $- + - + = 3$ variations.
" $x = +\infty$, " $+ + + + = 0$ "

Therefore, this equation has three real roots.

For $x = -5$, we find $- + - + = 3$ variations,
" $x = -4$, " $+ + - + = 2$ "
" $x = 1$, " $+ - - + = 2$ "
" $x = 2$, " $- - - + = 1$ "
" $x = 3$, " $+ + + + = 0$ "

Hence, there is one negative root between -4 and -5; one positive root between 1 and 2; and one positive root between 2 and 3.

For the positive root between 1 and 2, we have the following

OPERATION.

1	± 0	-15	$-21(1.769149632$ &c.$=x$.
	1	-14	-14
	2	-12	
	3	-941	-7
	37	-633	-6587
	44	-60204	
	51	-57072	-413
	516	-5659599	-361224
	522	-5611917	
	528	-561138629	-51776
	5289	-561085557	-50936391
	5298	-5610.6432764	
	5307		-839609
	53071		-561138629
	53072		
	53073		-278470371
	530734		-224425731056
			-54044.639944
			-50495
			-3549
			-3366
			-183
			-168
			-15
			-11
			3.

For the negative root, we have the following

OPERATION.

1	±0	−15	−21(−4.441621651 &c. $=x$.
	−4	1	−4
	−8	33	———
	−12	3796	−17
	−124	4308	−15184
	−128	436096	———
	−132	441408	−1816
	−1324	44154121	−1744384
	−1328	44167443	———
	−1332	4417543716	−71616
	−13321	4418343168	−44154121
	−13322	4418.36981764	———
	−13323		−27461879
	−133236		−26505262296
	−133242		———
	−133248		−956616704
	−1332482		−883673963528
			———
			−7294.2740472
			−4418
			———
			−2876
			−2651
			———
			−225
			−221
			———
			−4
			−4
			———
			0.

For the positive root between 2 and 3, we have this

OPERATION.

1	± 0	-15	$-21(2.672472018$ &c. $= x$.
	2	-11	-22
	4	-3	——
	6	96	1
	66	528	576
	72	58309	——
	78	63867	424
	787	6402724	408163
	794	6418752	——
	801	642195856	15837
	8012	642516528	12805448
	8014	6425.7264889	——
	8016		3041552
	80164		2568783424
	80168		——
	80172		462768576
	801727		449800854223
			——
			12967.721777
			12851
			——
			116
			64
			——
			52
			51
			——
			1.

Hence, the three roots of $x^3 - 15x = -21$, are
$x = -4.441621651$; 1.769149632 ; 2.672472018.

16. Find the three roots of the equation

$$10000x^3 - 4519x^2 + 665x = 32.$$

Applying Sturm's Theorem to this equation, we find

$$X = 10000x^3 - 4519x^2 + 665x - 32,$$
$$X_1 = 30000x^2 - 9038x + 665,$$
$$X_2 = 942722x - 125135,$$
$$X_3 = 5425404570000.$$

When $x=0$, we find $-+-+=3$ variations,
" $x=1$, " $++++=0$ "

Hence, the equation has three positive roots, each less than 1.

When $x=0.1$, we find $-+-+=3$ variations,
" $x=0.2$, " $++++=0$ "

Which shows that the first figure of each root is 0.1.

Again, when $x=0.11$, we find $-+-+=3$ variations.
" $x=0.12$, " $++-+=2$ "
" $x=0.13$, " $+--+=2$ "
" $x=0.14$, " $--++=1$ "
" $x=0.19$, " $-+++=1$ "

From this, we discover that the first two figures of the least root are 0.11; the first two figures of the next root are 0.13; the first two figures of the other root are 0.19.

For the first root we have the following:

10000	−4519	665	32 (0.119503816&c.
	−3519	3131	3131
	−2519	612	
	−1519	4701	69
	−1419	3382	4701
	−1319	23659	
	−1219	14308	2199
	−1129	138360	212931
	−1039	133665	
	−949	1336.369809	6969
	−944		6918
	−939		
	−934		51
	−93397		4009109427
			10908.90573
			10691
			217
			134
			83
			80
			3.

For the second root we have the following

OPERATION.

10000	—4519	665	32 (0.137139216&c.
	—3519	3131	3131
	—2519	612	
	—1519	2463	69
	—1219	—294	7389
	—919	—6783	
	—619	—10136	—489
	—549	—101768	—47481
	—479	—102175	
	—409	—10229671	—1419
	—408	—10241833	—101768
	—407	—1024.547809	
	—406		—40132
	—4057		—30689013
	—4054		
	—4051		—9442987
	—40501		—9220930281
			—2220.56719
			—2049
			—171
			—102
			—69
			—61
			—8.

For the third root we have the following

10000	—4519	665	32 (0.195256967 &c
	—3519	3131	3131
	—2519	612	
	—1519	549	69
	—619	3078	4941
	281	36935	
	1181	43340	1959
	1231	436066	184675
	1281	438736	
	1331	43940475	11225
	1333	44007375	872132
	1335	440.1540636	
	1337		250368
	13375		219702375
	13380		
	13385		30665625
	133856		26409243816
			4256.381184
			3962
			294
			264
			30
			30
			0.

Hence, the three roots of the equation

$$10000x^3 - 4519x^2 + 665x = 32,$$

are 0.119503816 ; 0.137139216 ; 0.195256967.

17. Find the three roots of the equation $x^3+2x^2-3x=9$.

Applying to this equation the Theorem of Sturm, we find

$$X = x^3+2x^2-3x-9,$$
$$X_1 = 3x^2+4x-3,$$
$$X_2 = 26x+75,$$
$$X_3 = -7047.$$

When $x = -\infty$, we find $- + - - = 2$ variations,
" $x = +\infty$, " $+ + + - = 1$ "
" $x = 1$, " $- + + - = 2$ "
" $x = 2$, " $+ + + - = 1$ "

Then this equation has but one real root, which lies between 1 and 2, the other roots being imaginary.

We find the real root by the following :

1	2	3	9(1.939465 &c. $= x$.
	3	0	0
	4	4	-
	5	931	9
	59	1543	8379
	68	156619	——
	77	158947	621
	773	15964891	469857
	776	16035163	——
	779	1603.828996	151143
	7799		143684019
	7808		——
	7817		7458981
	78174		6415315984
			——
			10436.65016
			9623
			——
			813
			801
			——
			12.

Dividing x^3+2x^2-3x-9 by $x-1.939465$, we find $x^2+3.939465x+4.640455$ for the quotient.

Therefore, solving the quadratic

$$x^2+3.939465x=-4.640455,$$

we find the following imaginary roots,

$$x=\begin{cases}-1.96973+0.87213\sqrt{-1},\\ -1.96973-0.87213\sqrt{-1}.\end{cases}$$

18. Find the three roots of the equation $x^3 - 5x^2 + 8x = 1$.

By Sturm's Theorem we have already found, page 381,

$$X = x^3 - 5x^2 + 8x - 1,$$
$$X_1 = 3x^2 - 10x + 8,$$
$$X_2 = 2x - 31,$$
$$X_3 = - 2295.$$

When $x = -\infty$, we find $- + - - = 2$ variations,
" $x = +\infty$, " $+ + + - = 1$ "
" $x = 0$, " $- + - - = 2$ "
" $x = 1$, " $+ + - - = 1$ "

Hence, this equation has one positive root which lies between 0 and 1, and two imaginary roots.

Its real root is found by the following

OPERATION.

1	—5	8	1 (0.1362934&c.$=x$.
	—49	751	751
	—48	703	—
	—47	68899	249
	—467	67507	206697
	—464	6723076	—
	—461	6695488	42303
	—4604	669.456964	40338456
	—4598		—
	—4592		1964544
	—45918		1338913928
			—
			6256.30072
			6025
			—
			231
			201
			—
			30
			27
			—
			3.

By dividing $x^3 - 5x^2 + 8x - 1$ by $x - 0.1362934$,

we find the quadratic $x^2-4.8637066x=-7.3371089$, which gives the following imaginary roots:

$$x=\begin{cases}2.43185+1.19298\sqrt{-1},\\ 2.43185-1.19298\sqrt{-1}.\end{cases}$$

19. Find one of the roots of $x^3-2x=5$.

Ans. $x=2.09455148$&c.

20. Find one of the roots of $2x^3+3x=90$.

Ans. $x=3.41639726$&c.

21. Find one of the roots of $x^3+x^2+x=100$.

Ans. $x=4.26442997$&c.

22. Find one of the roots of $x^3+x=500$.

Ans. $x=7.89500828$&c.

23. Find one of the roots of $x^3+10x^2+5x=2600$.

Ans. 11.00679933&c.

SOLUTION OF EQUATIONS ABOVE THE THIRD DEGREE.

(316.) It is obvious that the above method which we have employed for cubic equations, will apply equally well to equations of a superior degree. By carefully studying the preceding method, we shall be able to deduce, for equations of the nth degree, this general

RULE.

1. *Having found the first figure of the root, multiply it into the first coefficient and add the product to the second coefficient, which sum multiply by the same figure and add the product to the third coefficient, and this sum must be multiplied by the same figure and the product added to the fourth coefficient; and so continue to multiply the last result by this same figure and to add the product to the next*

succeeding coefficient, until the last coefficient is reached, which last sum must be multiplied by the same figure and the product subtracted from the term constituting the right-hand member of the equation; the remainder we will call the FIRST DIVIDEND.

Again, multiply the first coefficient by the same figure, and add the product to the number under the second coefficient, which sum must be multiplied by the same figure, and the product added to the term under the third coefficient; and thus we must continue to multiply and add, until we have obtained the second term under the last coefficient, which result we shall call the FIRST TRIAL DIVISOR.

Again, multiply the first coefficient by the same figure of the root, and add the product to the last term under the second coefficient, which result must be multiplied by the same figure, and the product added to the last number under the third coefficient; and thus we must continue to multiply and add until we reach the coefficient next to the last, when we must again begin with the first coefficient and multiply and add as before, until we reach the $n-2$th coefficient; then, again, commencing with the first coefficient, we must multiply and add until we reach the $n-3$d coefficient; we must continue this process, until we have thus obtained n terms under the second coefficient, $n-1$ terms under the third coefficient, $n-2$ terms under the fourth coefficient, $n-3$ terms under the fifth coefficient, and so of the rest.

II. *Find the second figure of the root, by dividing the* FIRST DIVIDEND *by the* FIRST TRIAL DIVISOR, *proceed with this second figure, precisely as was done with the first figure, observing to keep the work so that units shall stand under units, tens under tens, &c., &c.*

EXAMPLES.

1. Find one of the roots of the equation

$$3x^4 + x^3 + 4x^2 + 5x = 375.$$

OPERATION.

3	1	4	5	375(3.13364 &c. $= x$.
	10	34	107	321
	19	91	380	
	28	175	397873	54
	37	17873	416122	397873
	373	17249	421744861	
	376	18628	427402264	142127
	379	1874287	427.971813721	1265234583
	382	1885801		
	3829	1897342		156035417
	3838	189849907		1283915441163
	3847			
	3856			2764.38728837
	38569			2567
				197
				171
				26.

2. Find the four roots of the equation

$$x^4 - 80x^3 + 1998x^2 - 14937x = -5000.$$

FIRST OPERATION.

1	-80	1998	-14937	-5000(34.83228 &c. $= x$.
	-50	498	3	90
	-20	-102	-3057	
	10	198	-1561	-5090
	40	374	703	-6244
	44	566	1358552	
	48	774	2050968	1154
	52	81944	2078388507	10668416
	56	86552	2105862348	
	568	91224	2107697832088	671584
	576	9140169	210.9533553472	6235165521
	584	9157947		
	592	9175734		480674479
	5923	917742044		4215395664176
	5926	917860692		
	5929			591.349125824
	5932			422
				169
				168
				1.

SECOND OPERATION.

1	—80	1998	—14937	—5000(32.06029&c.=x.
	—50	498	3	90
	—20	—102	—3057	———
	10	198	—2493	—5090
	40	282	—1753	—4986
	42	370	—1725106984	———
	44	462	—169.7040736	—104
	46	4648836		—10350641904
	48	4677708		———
	4806			—49.358096
	4812			—34
				———
				—15
				—15
				———
				0.

THIRD OPERATION.

1	—80	1998	—14937	—5000(12.75644&c.=x.
	—70	1298	—1957	—19570
	—60	698	5023	———
	—50	198	5267	14570
	—40	122	5367	10534
	—38	50	5339063	———
	—36	—18	5296132	4036
	—34	—3991	5291946125	37373441
	—32	—6133	5287.687500	———
	—313	—8226		2986559
	—306	—837175		26459730625
	—299	—851725		———
	—292			34058.59375
	—2915			31726
	—2910			———
				2332
				2115
				———
				217
				212
				———
				5.

FOURTH OPERATION.

1	—80	1998	—14937	—5000	(0.35098&c.=x.
	—797	197409	—14344773	—43034319	
	—794	195027	—13759692		
	—791	192654	—13663561875	—6965681	
	—788	19226025	—135.67638500	—68317809375	
	—7875	19186675			
	—7870			—133.9000625	
				—122	
				—11	
				—10	
				—1.	

Hence, the four roots, true to 5 decimal places, are

34.83228 ; 32.06029 ; 12.75644 ; 0.35098.

3. Find the four roots of the equation

$$x^4+4x^3-4x^2-11x+4=0.$$

By Sturm's Theorem we have

$$X=x^4+4x^3-4x^2-11x+4=0,$$
$$X_1=4x^3+12x^2-8x-11,$$
$$X_2=20x^2+25x-27,$$
$$X_3=227x+31,$$
$$X_4=1547988.$$

When	$x=-5$,	we find	$+-+-+=4$	variations.
"	$x=-4$,	"	$-++-+=3$	"
"	$x=-2$,	"	$-++-+=3$	"
"	$x=-1$,	"	$++--+=2$	"
"	$x=0$,	"	$+--++=2$	"
"	$x=1$,	"	$--+++=1$	"
"	$x=2$,	"	$+++++=0$	"

Hence, there is one negative root between —5 and —4 ; one negative root between —2 and —1 ; one positive root between 0 and 1 ; and one positive root between 1 and 2.

These roots when found are

$$x = -4.2834,$$
$$x = -1.6908,$$
$$x = 0.3373,$$
$$x = 1.6369.$$

4. Find one of the roots of the equation

$$2x^5 + 5x^4 + 6x^3 + 2x^2 - 3x = 300.$$

OPERATION.

2	5	6	2	−3	300(2.223349&c.
	9	24	50	97	194
	13	50	150	397	—
	17	84	318	4658432	106
	21	126	344216	5401360	9316864
	25	13108	371464	54819013632	———
	254	13624	399760	55630349560	1283136
	258	14148	402706816	5575306659296506	1096338027264
	262	14680	405664464	558.759248434410	———
	266	1473408	408632960		186755727736
	270	1478824	409080108854		1678691397779686
	2704	1484248	409527502616		———
	2708	1489680			1949.65276690314
	2712	149049618			1676
	2716	149131254			—
	2720				273
	27206				223
	27212				—
					50
					50
					—
					0.

5. Find one of the roots of the equation

$$2x^5 - 7x^3 + 10x = 9$$

OPERATION.

2	0	−7	0	10	9(1.630101025 &c.
	2	−5	−5	5	5
	4	−1	−6	−1	–
	6	5	−1	54992	4
	8	13	10832	217760	329952
	10	1972	27128	2326581362	———
	112	2716	48320	2479627610	70048
	124	3532	49660454	248015150553963002	6979744086
	136	4420	51015416	24806.7544722052010	———
	148	446818	52384940		25055914
	160	451654	52389553963002		248015150553963002
	1606	456508	52394168089008		———
	1612	461380			2543.989446036998
	1618	4613963002			2481
	1624	4614126006			———
	1630				62
	163002				50
	163004				—
					12
					12
					—
					0.

6. Find one of the roots of the equation

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x = 54321.$$

OPERATION.

1	2	3	4	5	54321(8.4144 &c.
	10	83	668	5349	42792
	18	227	2484	25221	———
	26	435	5964	277224336	11529
	34	707	6253584	303424400	1108897344
	42	72396	6550016	3041105122401	———
	424	74108	6853360	304.7974011605	44002656
	428	75836	6861122401		3041105122401
	432	77580	6868889204		———
	436	7762401			1359.160477599
	440	7766803			1219
	4401				———
	4402				140
					122
					—
					18.

7. Find a root of the equation

$$26x^4 + 281x^3 - 576x^2 + 298x = 25.$$

Ans. $x = 0.77933994$ &c.

8. Find a root of the equation

$$x^5 - 5x^3 + 5x = 1.$$

Ans. $x = 0.20905692$ &c.

9. Find a root of the equation

$$x^6 + 2x^5 + 3x^4 + 4x^3 + 5x^2 + 6x = 654321.$$

Ans. $x = 8.95697957$ &c.

10. Find a root of the equation

$$2x^7 - 6x^6 - 5x^5 + 20x^4 + 2x^3 - 18x^2 + 4x = 4.$$

Ans. $x = 2.62599736$ &c.